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MATHEMATICAL KEY.

NEW COMBINATIONS

IN RESPECT TO THE

Binomial Theorem and Logarithms;

AND A

NEW DISCOVERY OF ONE GENERAL ROOT THEOREM FOR THE SOLUTION OF EQUATIONS OF ALL DEGREES:

THE EQUATION, $X^x = A$, OR ANY SIMILAR ONE NOT EXCEPTED.

BY JOSEPH B. MOTT.

DESIGNED FOR SUCH AS HAVE FIRST STUDIED SOME SIMPLE WORK ON ALGEBRA, AND DESIRE TO HAVE A MORE PERFECT KNOWLEDGE OF THAT USEFUL BRANCH OF MATHEMATICS.

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PREFACE.

THE discovery of a general theorem which will develop the roots of equations of all degrees, has seemed to the author to be sufficient apology for offering the subsequent pages to the world. They might have been much extended : many new facts might have been presented, and various examples introduced, which would have greatly enhanced their utility. Indeed the more simple parts of algebra should be so connected with the results here given, as to form one complete chain of reasoning from beginning to end ; but the author's health and personal circumstances would not permit him to devote a sufficient portion of time to the accomplishment of such an object. And as an undue extension of plan is often a cause of failure in the undertaking to carry it out, whereby many useful discoveries may be long hindered from seeing the light, I have thought it best to present these researches as they are, without any modification or enlargement, and run the risk as to what I may be able to do hereafter.

Few know or can realize the difficulties that must be encountered by one who undertakes to get up anything original: first, the great variety of experiments that must be made, take up a great deal of time, often bewilder the mind, and frequently amount to nothing but to show what cannot be done in that way ; then again the mental faculties are apt to be so intensely occupied, that the loss of health is a common result. Those who spend a large portion of their time in this way, have just so much less left to devote to the acquirement of property, or the means of living, for the want of a sufficiency of which they often fail of accomplishing their object at last ; or if a partial success attends their efforts, yet what is brought to view is new and unpopular, and perhaps but

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little noticed during the life of the discoverer. On the other hand, however, inventions and improvements follow each other so rapidly at the present day, and so many things have been brought to light that were scarcely dreamed of a short time ago, that public expectation seems more than formerly to be on the lookout for new discoveries and developments. Perhaps, then, among the denizens of this busy and curious world, some may be found to take an interest in the examination of the following pages.

The method of developing the coefficients in the binomial theorem, and finding the logarithmic series, etc., I think will be acknowledged as more simple than any hitherto in use; and the application of the theorems given to the finding of converging series, as often as desired, will serve more fully to exhibit their facility and usefulness.

As it respects *the general root theorem*, I claim both the discovery and invention of it as my own. If any other person has made a similar discovery, it must be a late occurrence, and is wholly unknown to me. DAVIES and other late authors say expressly that no direct formula has appeared for the solution of equations of a degree higher than the fourth, except when the roots are part or all of them rational; but it will be seen in the following pages that equations of all degrees, even many cases of infinity, have a direct solution by one general rule, that is less tedious when applied to numerical equations than any rule of approximation, as well as more perfect.

One great obstacle to the successful prosecution of algebraical researches heretofore has been the tedious methods of resolving the higher equations: the design of this work is to obviate that difficulty, and bring all equations nearly on a level. The reason why mathematicians have not long ago discovered a general theorem for this purpose, is because they have generally commenced their investigations by placing the

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terms of the highest powers of the unknown quantity first at the left hand; in which position, no direct formula could ever possibly be discovered for cases of infinity. Indeed all that has ever been done in this way consists merely in proposing a new formula (if not more than one) for every different degree of equation, until arriving at those of the fourth degree, where all perfect rules have hitherto suddenly stopped; and even the formulas which have been used for cubic and biquadratic equations are so tedious in most cases, when numerically applied, that the less perfect rules of approximation are generally preferred.

It is easy for a mere tyro in algebra to propose equations that will puzzle an adept to solve by former rules, but will not be so easy after these researches shall become known. There are various forms to which equations may be reduced before applying the root theorem, that will much simplify the result in many cases : these forms are not all presented, but only those which are of the highest importance in respect to solution, though many others might have been added with propriety.

My main object has been to present something new and useful : how far I have succeeded, I leave for the competent to judge. The methods given for the resolution of quadratics in the popular works on algebra are not to be overlooked, although not introduced here; but all the methods for solving the higher equations may be dispensed with beside the root theorem, except cases in which some of the roots are rational, or where the equation is solvable by the binomial or logarithmic theorem. The particular utility of the root theorem consists in the determination of the irrational roots of degrees higher than the second.

JOSEPH B. MOTT.

Dowagiac, Cass Co. Mich.

March, 1855.

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N. B. It is possible that some slight errors have been overlooked in the writing or printing of the results of examples given in the solution of equations, as many of them were noted down barely by inspection; but the theorems have been guarded in this particular, and are sufficient to rectify all mistakes, should there be any.

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INTRODUCTORY REMARKS.

If the binomial theorem be well understood, even in its most simple form as first presented in the following paragraphs, it will, I trust, by the method here proposed, be found a comparatively easy task to accomplish what has not hitherto been effected with equal brevity and perspicuity, if at all. I am aware that in some developments I have presented things in a manner different from the customary proceeding of mathematical writers, and that my reasons for so doing may not always appear at first sight; but the propriety of my assumptions will generally become manifest in the results. A disposition to experiment in substitution is probably the main reason for the course I have frequently pursued.

In my opinion, the methods of algebra may yet be made to cover the whole ground of the calculus, and even to go beyond it. If it be proposed to give a plain illustration of the principle of maximum and minimum, or of the distinguishing properties of a spiral, etc., I should prefer a resort to algebra rather than to the differential and integral calculus; though it is true I have not had time to examine all the mysterious developments that are reported in the higher mathematics, so as to see whether they can be referred to algebraical methods or not. Nevertheless I have not a doubt but that the blind methods of reasoning (so at least they appear to many minds) introduced in the calculus might be wholly dispensed with.

As some may wish to study these few pages who have not yet paid any attention to the nature and use of logarithms, I present the following as an

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INTRODUCTION TO LOGARITHMS.

ART. 1. First take the equation $b^x = N$; in which, if we consider b as the base of a system of logarithms, then x is the logarithm of the number N; that is, $\log N = x$, where log stands for logarithm.

Also if 10 be the base in the equations $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, &c., then $\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$, &c.

In like manner, in the equations

$$b^x = N, \quad b^{x'} = N', \quad b^{x''} = N'', \quad \&c., \ldots\ldots\ldots (1)$$

$$\log N = x, \quad \log N' = x', \quad \log N'' = x'', \quad \&c.$$

Now if we take the product of the equations

$$10^1 = 10, \quad 10^2 = 100, \quad 10^3 = 1000, \quad \text{we have}$$

$$10^1 \times 10^2 \times 10^3 = 10 \times 100 \times 1000 = 1000000 = 10^6 = 10^{1+2+3};$$

but according to our former position, in the equation

$$10^6 = 1000000, \ \log 1000000 = 6 = 1+2+3 = \log 10 + \log 100 + \log 1000.$$

Again, if we take product of equations (1), then

$$b^x \times b^{x'} \times b^{x''} = N \times N' \times N'', \quad \text{or}$$

$$b^{x+x'+x''} = N \times N' \times N'';$$

therefore $\log (N \times N' \times N'') = x + x' + x'' = \log N + \log N' + \log N''$.

Hence we come to the conclusion that the logarithm of the product of any number of factors is equal to the sum of the logarithms of the factors. .....(THEOREM 1.)

Now if we take the logarithms of 2 and 3 as hereafter computed, we have $\log 2 = .301030$ nearly,

and $\log 3 = .477121 +$;

then by the above theorem,

$$\log 4 = \log (2 \times 2) = \log 2 + \log 2 = 2 \log 2 = 2 \times .301030 = .602060.$$

Also by former considerations,

$$\log 10 = 1 = \log (2 \times 5) = \log 2 + \log 5 = 1, \quad \text{or}$$

$\log 5 = 1 - \log 2$; that is, $\log 5 = 1 - .301030 = .698970$.

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$\log 6 = \log(2\times3) = \log 2+\log 3 = .301030+.477121=.778151.$

$\log 8 = \log(2\times2\times2) = 3\log 2 = 3\times.301030 = .903090.$

$\log 9 = \log(3\times3) = 2\log 3 = 2\times.477121 = .954242.$

$\log 12=\log(3\times4) = \log 3+\log 4 =.477121+.602060=1.079181.$

We also have in the equation $p = ab$,

$$\log p = \log(ab) = \log a + \log b. \quad \ldots\ldots\ldots(1)$$

Or by division, $\frac{p}{a} = b$: then $\log b = \log \frac{p}{a}$;

but from equation (1), $\log b = \log p - \log a$;

therefore, $$\log \frac{p}{a} = \log p - \log a:$$

that is, the logarithm of a fraction is equal to the logarithm of the numerator, minus the logarithm of the denominator. (THEOREM 2.)

Or, for a more general theorem for fractions, let us resume the equation $\log \frac{p}{a} = \log p - \log a$, and make $p = qrs$ and $a = def$: then $\log \frac{qrs}{def} = \log qrs - \log def$; but by theorem 1, $\log qrs = \log q + \log r + \log s$, and $\log def = \log d + \log e + \log f$; therefore,

$$\log \frac{qrs}{def} = \log q + \log r + \log s - \log d - \log e - \log f:$$

that is, the logarithm of a fraction composed of factors is equal to the sum of the logarithms of the factors constituting the numerator, diminished by the sum of the logarithms of the factors constituting the denominator.....(THEOREM 2m.)

Therefore by the two last theorems, etc., we have

1. $\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1-.301030 = .698970.$
2. $\log \frac{2}{3} = \log 2 - \log 3 = .301030 - .477121 = -.176091.$
3. $\log \frac{3\times5\times10}{2\times4\times8} = \log 3 + \log 5 - 6\log 2$

(as $2\times4\times8 = 2\times2\times2\times2\times2\times2$),

$= .477121 + .698970 + 1-6\times.301030$

$= .369911 = \log \frac{150}{64}.$

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4. In the equations $xy = a$ and $\frac{x}{y} = b$, we have

$$\log x + \log y = \log a \text{ and } \log x - \log y = \log b,$$

or $\log x = \frac{1}{2}(\log a + \log b)$ and $\log y = \frac{1}{2}(\log a - \log b)$.

5. $\log 1 = \log \frac{n}{n} = \log n - \log n = 0$:

$\log \frac{1}{27} = 0 - 3 \log 3$ (since $27 = 3 \times 3 \times 3$),

or $\log \frac{1}{27} = -1.431363$.

6. $\log \frac{1}{30} = -\log 3 - 1 = -1.477121$.

7. $\log \frac{1}{2 \times 3 \times 6} = -2 \log 2 - 2 \log 3 = -1.556302$

$= \log \frac{1}{36}$.

Thirdly, as $\log (aaaa) = \log a^4 = 4 \log a$, therefore

$$\log a^n = n \log a;$$

and if $n = \frac{r}{m}$, then $\log a^{\frac{r}{m}} = \frac{r}{m} \log a$:

that is, the logarithm of any power or root of a number is equal to the logarithm of the number multiplied by the exponent. ---------- ----------(THEOREM 3.)

1. $\log 81 = \log 3^4 = 4 \log 3 = 4 \times .477121 = 1.908484$.

2. $\log 1024 = \log 2^{10} = 10 \log 2 = 10 \times .301030 = 3.010300$.

3. $\log \sqrt{2} = \log 2^{\frac{1}{2}} = \frac{1}{2} \log 2 = \frac{1}{2} \times .301030 = .150515$.

4. $\log \sqrt[3]{3} = \log 3^{\frac{1}{3}} = \frac{1}{3} \log 3 = \frac{1}{3} \times .477121 = .159040$.

5. $\log \sqrt[10]{5} = \log 5^{\frac{1}{10}} = \frac{1}{10} \log 5 = \frac{1}{10} \times .698970 = .069897$.

6. If $2^x = 100$, then $x \log 2 = \log 100 = 2$, or $x = \frac{2}{\log 2}$.

7. If $3^{x+y} = 9$ and $5^{x-y} = 6$, then

$$x+y = \frac{\log 9}{\log 3} = 2 \text{ and } x-y = \frac{\log 6}{\log 5},$$

or $x = \frac{1}{2}\left(2 + \frac{778151}{698970}\right)$ and $y = \frac{1}{2}\left(2 - \frac{778151}{698970}\right)$.

8. If $a^x = b$, then $x \log a = \log b$, or $x = \frac{\log b}{\log a}$.

9. If $c^{\frac{y}{z}} = a$, then $\frac{y}{z} \log c = \log a$, or $\frac{y}{z} = \frac{\log a}{\log c}$:

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that is, the logarithm of any power or root of a number, divided by the logarithm of the number, is equal to the exponent. ----------(THEOREM 4.)

10. If $x^y = a$ and $bx^c = d$, then

$$y \log x = \log a \text{ and } \log b + c \log x = \log d, \text{ or}$$

$$\log x = \frac{1}{y} \log a \text{ and } \log x = \frac{1}{c}(\log d - \log b) = \frac{1}{y} \log a;$$

therefore, $$y = \frac{c \log a}{\log d - \log b}.$$

11. Given $wxy = a$, $wxz = b$, $wyz = c$ and $xyz = d$, to find the logarithms of w, x, y and z.

ANS. $\log w = \frac{1}{3}(\log a + \log b + \log c + \log d) - \log d$,

$\log x = \log w + \log d - \log c$, $\log y = \log w + \log d - \log b$,

and $\log z = \log w + \log d - \log a$.

12. Given $xyz = 4$, $\frac{x}{y^2 z} = 8$ and $x^3 y^2 z^{\frac{1}{2}} = 2$, to find the logarithms of x, y and z.

ANS. $\log x = \frac{15}{11} \log 2$, or $x = 2^{\frac{15}{11}}$;

$\log y = -\frac{25}{11} \log 2$, or $y = \frac{1}{2^{\frac{25}{11}}}$;

$\log z = \frac{32}{11} \log 2$, or $z = 2^{\frac{32}{11}}$.

Before much of the utility of logarithms can appear, we must have a table of them, in which the logarithm of any given number, and conversely the number answering to any given logarithm, can be readily found to a considerable degree of exactness by inspection: the method of computing such a table will be presented hereafter. Having such table, let it be required to find how long it would take any sum s of money to double itself at compound interest, at a given rate r.

By the arithmetical rule, the interest of the

First year $= rs$, and the amount $= s + rs = s(1+r)$;

Second year $= rs(1+r)$,
amount $= s(1+r) + rs(1+r) = s(1+r)(1+r) = s(1+r)^2$;

Third year $= rs(1+r)^2$,
amount $= s(1+r)^2 + rs(1+r)^2 = s(1+r)(1+r)^2 = s(1+r)^3$;

Fourth year $= rs(1+r)^3$,
amount $= s(1+r)^3 + rs(1+r)^3 = s(1+r)(1+r)^3 = s(1+r)^4$,

&c. &c.

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Therefore if A = amount, s = sum, r = rate and t = time, the general equation $A = s(1+r)^t$ will express the amount of any sum of money at a given rate per cent compound interest, for any length of time.

If the sum be at interest just long enough to double itself, we should have $A = 2s$, or $2s = s(1+r)^t$, or $(1+r)^t = 2$: then, by theorem 3, $t\log(1+r) = \log 2$, or $t = \frac{\log 2}{\log(1+r)}$.

If the sum remain at interest until it trebles itself, or until it becomes ten time its first value, then

$$t = \frac{\log 3}{\log(1+r)}, \text{ or } t = \frac{\log 10}{\log(1+r)} = \frac{1}{\log(1+r)}.$$

Hence any sum, at 7 per cent compound interest, will double in the time denoted by the equation

$$t = \frac{\log 2}{\log 1.07} = \frac{.301030}{.029384} = 10.2447 \text{ years} = \overset{\text{Yrs.}}{10}\ \overset{\text{mos.}}{2}\ \overset{\text{das.}}{28} \text{ nearly.}$$

At 6 per cent,

$$t = \frac{\log 2}{\log 1.06} = \frac{.301030}{.025306} = 11.8956 \text{ years} = 11\ 10\ 23 \text{ nearly.}$$

One arithmetician gives 10 years 87 days as near the time at 7 per cent. Others give 11 years 10 months 22 days for 6 per cent : 23 days come nearer to the truth than 22 ; $22\frac{5}{6}$, nearer than either.

EXPANSION OF BINOMIALS.

ART. 2. Let it be required to expand the binomial $(1+r)^s$.

First let us find the 2d, 3d, 4th, &c. powers of $(1+r)$.

Thus $1+r$	and $1+2r+r^2$
into $1+r$	into $1+r$
gives $1+2r+r^2 = (1+r)^2$,	gives $1+3r+3r^2+r^3 = (1+r)^3$.

And in like manner we may find

$(1+r)^4 = 1+4r+6r^2+4r^3+r^4$, ...(1)

$(1+r)^5 = 1+5r+10r^2+10r^3+5r^4+r^5$, ...(2)

$(1+r)^6 = 1+6r+15r^2+20r^3+15r^4+6r^5+r^6$, ...(3)

$(1+r)^7 = 1+7r+21r^2+35r^3+35r^4+21r^5+7r^6+r^7$, ...(4)

$(1+r)^8 = 1+8r+28r^2+56r^3+70r^4+56r^5+28r^6+8r^7+r^8$, ..(5)

&c. &c.

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ART. 3. We now propose to find the general law that governs the coefficients and exponents in equations (1), (2), (3), etc. By inspection, it is seen that the exponents commence in the second term with unity, and increase in each succeeding one by the same to the last term.

The terms without the coefficients are 1, r, r^2, r^3, etc.; in which we see that each term (after the first) is a multiple of the preceding term, and has one more factor. Now let us make a similar supposition in regard to the coefficients, by assuming A = coefficient of r, A·B = coefficient of r^2, A·B·C of r^3, A·B·C·D of r^4, &c.*. Then we have

$$(1+r)^s = 1+Ar+A\cdot Br^2+A\cdot B\cdot Cr^3+A\cdot B\cdot C\cdot Dr^4+\&c. \quad ..(S)$$

Now if we make $s = 8$, and compare equations (5) and (S), (as they become equal) we have

$$A=s=8,\ A\cdot B=28,\ A\cdot B\cdot C=56,\ A\cdot B\cdot C\cdot D=70,\ A\cdot B\cdot C\cdot D\cdot E=56,$$
$$\&c.$$

Hence
$$B = \frac{28}{A} = \frac{28}{8} = \frac{7}{2} = \frac{A-1}{2} = \frac{s-1}{2};$$
$$C = \frac{56}{A\cdot B} = \frac{56}{28} = \frac{6}{3} = \frac{A-2}{3} = \frac{s-2}{3};$$
$$D = \frac{70}{A\cdot B\cdot C} = \frac{70}{56} = \frac{5}{4} = \frac{A-3}{4} = \frac{s-3}{4};$$
$$\&c. \qquad \&c.$$

If we substitute these values of A, B, C, etc. in equation (S), we get

$$(1+r)^s = 1+sr+s.\frac{s-1}{2}r^2+s.\frac{s-1}{2}.\frac{s-2}{3}r^3+s.\frac{s-1}{2}.\frac{s-2}{2}.\frac{s-3}{3}r^4$$
$$+\&c. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(k)$$

when $s = 8$.

To see whether this series will remain the same when s has a different value, let $s = 6$, and compare equations (3) and (S) : then we have

$$A=6=s,\ A\cdot B=15,\ A\cdot B\cdot C=20,\ A\cdot B\cdot C\cdot D=15,\ \&c.$$

* When dots are used for × to denote multiplication, they will (except when placed between fractions, or between a fraction and a whole number) be elevated, in order not to confound the decimal point.

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Consequently $B = \frac{15}{A} = \frac{15}{6} = \frac{5}{2} = \frac{A-1}{2} = \frac{s-1}{2}$;

$C = \frac{20}{A \cdot B} = \frac{20}{15} = \frac{4}{3} = \frac{A-2}{3} = \frac{s-2}{3}$;

$D = \frac{15}{A \cdot B \cdot C} = \frac{15}{20} = \frac{3}{4} = \frac{A-3}{4} = \frac{s-3}{4}$, &c.

Substituting these values of A, B, C, etc. in equation (S), we get

$$(1+r)^s = 1+sr+s.\frac{s-1}{2}r^2+s.\frac{s-1}{2}.\frac{s-2}{3}r^3+s.\frac{s-1}{2}.\frac{s-2}{3}.\frac{s-3}{4}r^4 + \&c. \quad \ldots\ldots\ldots\

Here, in theorem (K), let $r = \frac{b}{a}$ and $s = n$: then

$$a^n(1+\tfrac{b}{a})^n = a^n(1+n\tfrac{b}{a}+n.\tfrac{n-1}{2}\tfrac{b^2}{a^2}+n.\tfrac{n-1}{2}.\tfrac{n-2}{3}\tfrac{b^3}{a^3} +n.\tfrac{n-1}{2}.\tfrac{n-2}{3}.\tfrac{n-3}{4}\tfrac{b^4}{a^4}+\&c.),$$

or $$(a+b)^n = a^n+na^{n-1}b+n.\tfrac{n-1}{2}a^{n-2}b^2+n.\tfrac{n-1}{2}.\tfrac{n-2}{3}a^{n-3}b^3 +n.\tfrac{n-1}{2}.\tfrac{n-2}{3}.\tfrac{n-3}{4}a^{n-4}b^4+\&c.$$

3. Given $a^{\frac{m}{n}}\left(1+\frac{z}{a}\right)^{\frac{m}{n}}$ to expand by theorem (K).

Let $r = \frac{z}{a}$ and $s = \frac{m}{n}$: then the factors s, $\frac{s-1}{2}$, $\frac{s-2}{3}$, &c. will become $\frac{m}{n}$, $\frac{m-n}{2n}$, $\frac{m-2n}{3n}$, $\frac{m-3n}{4n}$, &c. Therefore,

$$a^{\frac{m}{n}}\left(1+\frac{z}{a}\right)^{\frac{m}{n}} = (a+z)^{\frac{m}{n}}$$
$$= a^{\frac{m}{n}}(1+\tfrac{m}{n}\tfrac{z}{a}+\tfrac{m}{n}.\tfrac{m-n}{2n}\tfrac{z^2}{a^2}+\tfrac{m}{n}.\tfrac{m-n}{2n}.\tfrac{m-2n}{3n}\tfrac{z^3}{a^3}+\&c\,)$$

Now if $m = 1$, this becomes

$$(a+z)^{\frac{1}{n}} = a^{\frac{1}{n}}(1+\tfrac{1}{n}\tfrac{z}{a}+\tfrac{1}{n}.\tfrac{1-n}{2n}\tfrac{z^2}{a^2}+\tfrac{1}{n}.\tfrac{1-n}{2n}.\tfrac{1-2n}{3n}\tfrac{z^3}{a^3}+\&c.);$$

and if moreover $a = 1$, we get

$$(1+z)^{\frac{1}{n}} = 1+\tfrac{1}{n}z+\tfrac{1}{n}.\tfrac{1-n}{2n}z^2+\tfrac{1}{n}.\tfrac{1-n}{2n}.\tfrac{1-2n}{3n}z^3 +\tfrac{1}{n}.\tfrac{1-n}{2n}.\tfrac{1-2n}{3n}.\tfrac{1-3n}{4n}z^4+\&c.$$

ART. 3. For more convenient reference, let us recapitulate :

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$$(1+r)^s = 1 + sr + s.\frac{s-1}{2}r^2 + s.\frac{s-1}{2}.\frac{s-2}{3}r^3 + s.\frac{s-1}{2}.\frac{s-2}{3}.\frac{s-3}{4}r^4 + \&c. \quad \text{(K)}$$

$$(a+b)^n = a^n + na^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 + n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3 + n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}a^{n-4}b^4 + \&c. \quad \text{(A)}$$

$$(1+z)^{\frac{1}{n}} = 1 + \frac{1}{n}z + \frac{1}{n}.\frac{1-n}{2n}z^2 + \frac{1}{n}.\frac{1-n}{2n}.\frac{1-2n}{3n}z^3 + \frac{1}{n}.\frac{1-n}{2n}.\frac{1-2n}{3n}.\frac{1-3n}{4n}z^4 + \&c. \quad \text{(B)}$$

$$(a+z)^{\frac{1}{n}} = a^{\frac{1}{n}}\left(1 + \frac{1}{n}\frac{z}{a} + \frac{1}{n}.\frac{1-n}{2n}\frac{z^2}{a^2} + \frac{1}{n}.\frac{1-n}{2n}.\frac{1-2n}{3n}\frac{z^3}{a^3} + \frac{1}{n}.\frac{1-n}{2n}.\frac{1-2n}{3n}.\frac{1-3n}{4n}\frac{z^4}{a^4} + \&c.\right). \quad \text{(C)}$$

$$(a+z)^{\frac{m}{n}} = a^{\frac{m}{n}}\left(1 + \frac{m}{n}\frac{z}{a} + \frac{m}{n}.\frac{m-n}{2n}\frac{z^2}{a^2} + \frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}\frac{z^3}{a^3} + \frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}.\frac{m-3n}{4n}\frac{z^4}{a^4} + \&c.\right). \quad \text{(D)}$$

Although theorem (K) is the key to all the others, and might be used in all cases, each one will be found more appropriate to some particular examples.

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If we examine theorem (B), or any other one having a fractional exponent, it may be seen that the series will not end. The reason of this is that the factors $\frac{1}{n}$, $\frac{1-n}{2n}$, $\frac{1-2n}{3n}$, etc., though produced ever so far, will never come to a term whose value is zero, except when $n = 1$, which would make $\frac{1}{n} = 1$, and thereby render the exponent integral contrary to the position taken. The same observation may be made in regard to formula (D), whenever the fraction $\frac{m}{n}$ can not be reduced to a whole number. Also in either of the formulas, when the exponent is supposed to be negative, the series will be infinite, and every other term will be negative. At other times, the terms will be all negative or all positive, according to the particular values given to the terms of the binomial and to the exponent.

The nature of binomials, and also of the more complicated developments, is such that a correct knowledge of the properties of positive and negative quantities is of the utmost importance when we engage in the substitution of numbers for letters.

Most algebraical authors enter into a separate examination of binomials with fractional and negative exponents, without deducing them from integrals by substitution as heretofore presented; but the same authors would consider it perfectly compatible to transform the equation $x^2 + ax = b$ into one whose roots should be reciprocals of those of the primitive, by making $x = \frac{1}{y}$; or into various other forms, by making $x = \frac{y}{z}$, $x = n+y$, $x = -ny$, $x = -\frac{n}{z}$, $x = \frac{m-n}{m+n}$, &c. &c. With equal consistency, then, may fractional and negative exponents be made to depend on integral and positive ones.

If any one wishes to determine the series that the binomial $(1+r)^{\frac{s}{n}}$ will produce without the aid of theorem (K), first find by the common rules the square root to three or four terms of $(1+r)$, and elevate it to the third power, or take the cube

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root of the same binomial and square it; that is, find $(1+r)^{\frac{3}{2}}$, $(1+r)^{\frac{2}{3}}$: then assume $(1+r)^{\frac{s}{n}} = 1+Ar+A\cdot Br^2+A\cdot B\cdot Cr^3+$&c. and make $\frac{s}{n}$ equal to $\frac{3}{2}$ in the former example, and to $\frac{2}{3}$ in the latter, and compare the coefficients in each development with the corresponding ones found by the first operation. Any other fractional power of the same binomial, whose development to three or four terms by the common methods would not be too cumbersome, might be treated in the same manner, to familiarize the student with the formation of the coefficients.

To study the properties of binomials having negative exponents, without resorting to theorem (K), we may first find a few terms of the series of such binomials as the following, $\frac{1}{(1+r)}$, $\frac{1}{(1+r)^2}$, $\frac{1}{(1+r)^{\frac{1}{2}}}$, &c. by the common methods, and then compare as before with the assumption

$$(1+r)^{-s} = 1+Ar+A\cdot Br^2+A\cdot B\cdot Cr^3+\&c.$$

EXAMPLES REQUIRING THE THEOREMS (K), (A), (B), (C) AND (D).

1. Expand $(1+2c)^3$ by theorem (K).

ANS. $(1+2c)^3 = 1+6c+12c^2+8c^3$.

2. Expand $(2+3y)^4$ by theorem (A).

ANS. $(2+3y)^4 = 16+96y+216y^2+216y^3+81y^4$.

3. By theorem (B),

$$(1+1)^{\frac{1}{2}} = 1+\tfrac{1}{2}-\frac{1}{2\cdot4}+\frac{3}{2\cdot4\cdot6}-\frac{3\cdot5}{2\cdot4\cdot6\cdot8}+\frac{3\cdot5\cdot7}{2\cdot4\cdot6\cdot8\cdot10}-\&c. = \sqrt{2}.$$

4. By theorem (C),

$$\left(\frac{196}{100}+\frac{4}{100}\right)^{\frac{1}{2}} = \tfrac{14}{10}\left(1+\frac{1}{2\cdot49}-\frac{1}{2\cdot4\cdot49^2}+\frac{3}{2\cdot4\cdot6\cdot49^3}-\frac{3\cdot5}{2\cdot4\cdot6\cdot8\cdot49^4}+\&c\right)$$
$$= \sqrt{2}.$$

5. By theorem (D),

$$(8+1)^{\frac{2}{3}} = 4\left(1+\frac{2}{3\cdot8}-\frac{2}{3\cdot6\cdot8^2}+\frac{2\cdot4}{3\cdot6\cdot9\cdot8^3}-\frac{2\cdot4\cdot7}{3\cdot6\cdot9\cdot12\cdot8^4}+\&c.\right) = 9^{\frac{2}{3}}$$
$$= \sqrt[3]{81}.$$

In examples 3 and 4, it is seen that the square root of 2 is expressed by series having very different terms.

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The series derived from $(1+1)^{\frac{1}{3}}$ is the one commonly chosen to exhibit the square root of 2 : it is, however, of little worth for computing the same; but the series of Ex. 4 converges very rapidly.

To obtain $\sqrt{2} = (1.96+.04)^{\frac{1}{2}}$, first find by trial that the square root of 2 is near 1.4 : then make 1.4^2 or 1.96 the first term of the binomial, and the second will be 2—1.96 = .04. A still more rapidly converging series can be found, by first taking a nearer value of $\sqrt{2}$.

5. Required the cube root of 2.

First let us try $2^{\frac{1}{3}} = (1+1)^{\frac{1}{3}}$: then by theorem (B),

$$(1+1)^{\frac{1}{3}} = 1+\tfrac{1}{3}-\frac{2}{3\cdot 6}+\frac{2\cdot 5}{3\cdot 6\cdot 9}-\frac{2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}+\&c. = \tfrac{4}{3} \text{ nearly.}$$

Now to get a more converging series, let $\frac{4^3}{3^3} = \frac{64}{27}$ be made the first term of the binomial, when the second will be $2-\frac{64}{27} = -\frac{10}{27}$; that is, $(\frac{64}{27}-\frac{10}{27})^{\frac{1}{3}} = \sqrt[3]{2}$. Then by theor. (C), $a = \frac{64}{27}$ or $a^{\frac{1}{3}} = \frac{4}{3}$, $z = -\frac{10}{27}$, $\frac{z}{a} = -\frac{5}{32}$, $n = 3$; and therefore

$$\sqrt[3]{2} = \left(\frac{64}{27}-\frac{10}{27}\right)^{\frac{1}{3}} = \tfrac{4}{3}\left(1-\frac{5}{3\cdot 32}-\frac{2\cdot 5^2}{3\cdot 6\cdot 32^2}-\frac{2\cdot 5\cdot 5^3}{3\cdot 6\cdot 9\cdot 32^3}-\&c.\right)$$
$$= 1.259921.$$

To find a series that will converge still faster, make the first term of the binomial equal to $1.26^3 = 2.000376$: we shall then have —.000376 for the second term; and by the theorem,

$$\sqrt[3]{2} = 1.26\left(1-\frac{376}{3\cdot 2000376}-\frac{2\cdot 376^2}{3\cdot 6\cdot 2000376^2}-\frac{2\cdot 5\cdot 376^3}{3\cdot 6\cdot 9\cdot 2000376^3}-\&c.\right).$$

6. Required the fifth root of 10.

By a few trials, 1.6 is found to be a near value of the root required : then we have $1.6^5 = 10.48576$ for the first term, and —.48576 for the second; that is,

$$10^{\frac{1}{5}} = (10.48576-.48576)^{\frac{1}{5}}.$$

Then by theorem (C),

$$10^{\frac{1}{5}} = 1.6\left(1-\frac{48576}{5\cdot 1048576}-\frac{4\cdot 48576^2}{5\cdot 10\cdot 1048576^2}-\frac{4\cdot 9\cdot 48576^3}{5\cdot 10\cdot 15\cdot 1048576^3}-\&c\right)$$
$$= 1.58489319246.$$

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7. Develop $(16+1)^{\frac{3}{4}}$ by theorem (D).

We have $a=16$, $z=1$, $m=3$, $n=4$, $a^{\frac{m}{n}}=8$, $\frac{z}{a}=\frac{1}{16}$, $\frac{m}{n}=\frac{3}{4}$, $\frac{m-n}{2n}=-\frac{1}{8}$, $\frac{m-2n}{3n}=-\frac{5}{12}$, &c. Hence

$$(16+1)^{\frac{3}{4}}=8\left(1+\frac{3}{4\cdot16}-\frac{3}{4\cdot8\cdot16^2}+\frac{3\cdot5}{4\cdot8\cdot12\cdot16^3}-\frac{3\cdot5\cdot9}{4\cdot8\cdot12\cdot16\cdot16^4}+\&c.\right)$$
$$=17^{\frac{3}{4}}.$$

ART. 4. The use of the binomial theorem in the raising of powers and extracting of roots has now become apparent. We next propose to apply it to the determination of unknown exponents, and to the development of series for the computation of logarithms.

Ex. The equation $B^x=1+a=N$ being given, to develop in certain cases.

From this equation, we have $B=(1+a)^{\frac{1}{x}}$. To get an easy development in terms of a, assume $(1+a)^{\frac{1}{x}}=(1+nx)^{\frac{1}{nx}}=B$: then $N=B^x=\left((1+nx)^{\frac{1}{nx}}\right)^x=(1+nx)^{\frac{1}{n}}=1+a$,

or $$1+nx=(1+a)^n;$$

whence $x=\frac{1}{n}(1+a)^n-\frac{1}{n}$, $N=(1+nx)^{\frac{1}{n}}$, $B=(1+nx)^{\frac{1}{nx}}$.

Developing the value of x by theorem (K), and those of N and B by theorem (B), we have

$$x=a+\frac{n-1}{1}\cdot\frac{a^2}{2}+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{a^3}{3}+\frac{n-1}{1}\cdot\frac{n-2}{2}\cdot\frac{n-3}{3}\cdot\frac{a^4}{4}+\&c.;$$

$$N=1+\frac{x}{1}+\frac{1-n}{1\cdot2}x^2+\frac{1-n}{1\cdot2}\cdot\frac{1-2n}{3}x^3+\frac{1-n}{1\cdot2}\cdot\frac{1-2n}{3}\cdot\frac{1-3n}{4}x^4+\&c.$$

$$B=1+\frac{1}{1}(1-nx)\frac{1}{1\cdot2}+(1-nx)(1-2nx)\frac{1}{1\cdot2\cdot3}$$
$$+(1-nx)(1-2nx)(1-3nx)\frac{1}{1\cdot2\cdot3\cdot4}+\&c.$$

As these equations are true for all values whatever of n, they will be so when $n=0$; on which supposition they will reduce to their simplest possible forms, namely:

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$$x = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \frac{a^5}{5} - \&c.; \quad \ldots\ldots\ldots\ldots\ldots\ldots (k)$$

$$N = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} + \&c. = 1+a, * \quad \ldots\ldots (N)$$

$$\text{or } a = x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4} + \frac{x^5}{2\cdot 3\cdot 4\cdot 5} + \&c.; \quad \ldots\ldots\ldots\ldots (k')$$

$$B = 1 + \frac{1}{1} + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \&c. = 2.718+.$$

Hence the equation $B^x = 1+a$ becomes

$$(2.718+)^{a-\frac{a^2}{2}+\frac{a^3}{3}-\frac{a^4}{4}+\&c.} = 1 + a,$$

which comprehends the napierian system of logarithms; that is, if 2.718+ be the base, then

$$l(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \frac{a^5}{5} - \&c.,$$

where l denotes napierian logarithm; and hereafter L will be used to denote the common logarithm, when placed before a quantity, instead of log; that is, the expression

l $(1+a)$ should be read: napierian logarithm of $1+a$, and
L $(1+a)$ .. .. .. : common logarithm of $1+a$, or so understood.

Resuming $l(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \frac{a^5}{5} - \&c.$: (1)

to find a more converging series, let $a = -b$, and we have

$$l(1-b) = -b - \frac{b^2}{2} - \frac{b^3}{3} - \frac{b^4}{4} - \frac{b^5}{5} - \&c. \quad \ldots\ldots (2)$$

Subtracting equation (2) from (1), we get

$$l(1+a) - l(1-b) = (a+b) - \frac{a^2-b^2}{2} + \frac{a^3+b^3}{3} - \frac{a^4-b^4}{4} + \frac{a^5+b^5}{5} - \&c.;$$

or, if we make $b = a$, and consider $l(1+a) - l(1-b) = l\left(\frac{1+a}{1-a}\right)$,

* The number answering to any napierian logarithm may be determined from equation (N), and a slight modification will give a formula for determining the number of any common logarithm.

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then $$\text{l}\left(\frac{1+a}{1-a}\right) = 2\left(a + \frac{a^3}{3} + \frac{a^5}{5} + \&c.\right) \ldots\ldots\ldots (\text{l})$$

This series might be converted into various other forms, but the present is about as simple as any, if we combine with it the equation $n = \frac{1+a}{1-a}$, which, reduced, gives $a = \frac{n-1}{n+1}$; or if we wish to compute the logarithm of a fraction, let $\frac{m}{n} = \frac{1+a}{1-a}$, whence we find $a = \frac{m-n}{m+n}$.

1. To compute the napierian logarithm of 2.

Make $2 = \frac{1+a}{1-a}$, whence $a = \frac{1}{3}$: then by theorem (l),

we find $$\text{l}\,2 = 2\left(\tfrac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7} + \&c.\right);$$

and by employing eight or nine terms of this series, we get $\text{l}\,2 = .69314718$.

2. For the computation of the napierian logarithm of 4, we have $\text{l}\,4 = \text{l}\,(2\times 2) = 2\,\text{l}\,2 = 1.38629436$.

3. To find the napierian logarithm of 5, since $\text{l}\,5 = \text{l}\,(4\times\frac{5}{4}) = \text{l}\,4 + \text{l}\,\frac{5}{4}$, it is only necessary to compute $\text{l}\,\frac{5}{4}$: then make $\frac{5}{4} = \frac{1+a}{1-a}$, whence $a = \frac{1}{9}$, and by theorem (l),

$$\text{l}\,\tfrac{5}{4} = 2\left(\tfrac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \frac{1}{7\cdot 9^7} + \&c.\right)$$
$$= 0.22314355,$$

and $$\text{l}\,4 = 1.38629436;$$

$$\text{l}\,4 + \text{l}\,\tfrac{5}{4} = \text{l}\,5 = 1.60943791.$$

4. To find $\text{l}\,10$, we have $\text{l}\,2 + \text{l}\,5 = \text{l}\,10 = 2.30258509$.

To compute the napierian logarithm of 10 by a somewhat different method (although the foregoing is nearly the same as that adopted by most authors), we may take the equality $(\frac{5}{4})^3 = \frac{125}{64}$: then $2 = \frac{125}{64} + \frac{3}{64} = \frac{125}{64}(1 + \frac{3}{125})$

$$= \frac{5^3}{4^3}(1 + \tfrac{3}{125}) = \frac{5^3}{4^3} \times \tfrac{128}{125},$$

or $$\text{l}\,2 = 3\,\text{l}\,\tfrac{5}{4} + \text{l}\,\tfrac{128}{125};$$

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also $l4 = 2l2 = 6l\frac{5}{4} + 2l\frac{128}{125}$; but $l\frac{5}{4} = l5 - l4$, or $l5 = l4 + l\frac{5}{4} = (6l\frac{5}{4} + 2l\frac{128}{125}) + l\frac{5}{4} = 7l\frac{5}{4} + 2l\frac{128}{125} = l5$; and as $l10 = l2 + l5$, therefore

$l10 = (7l\frac{5}{4} + 2l\frac{128}{125}) + (3l\frac{5}{4} + l\frac{128}{125}) = 10l\frac{5}{4} + 3l\frac{128}{125}$; but by formula (l),

$$l\tfrac{5}{4} = 2(\tfrac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \frac{1}{7\cdot 9^7} + \&c.),$$

and $l\frac{128}{125} = 2(\frac{3}{253} + \frac{3^3}{3\cdot 253^3} + \frac{3^5}{5\cdot 253^5} + \frac{3^7}{7\cdot 253^7} + \&c.)$: hence,

$$l10 = 20(\tfrac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \&c.) + 6(\tfrac{3}{253} + \frac{3^3}{3\cdot 253^3} + \frac{3^5}{5\cdot 253^5} + \&c.)$$

$$= 2.302585092994045684017991454684. \quad \text{Let } m = \frac{1}{l10} = 0.434294481903251827651128918916 6 = \text{modulus}:$$

$$2m = 0.868588963806503655302257837833 2.$$

Art. 5. From the foregoing data respecting napierian logarithms, we may proceed to compute common logarithms. For this purpose, let us first find the value of the unknown exponent x in the equation $b^x = 1+a$. Taking the napierian logarithm of both members, we have

$$x\,lb = l(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \&c.,$$

or $$x = \frac{1}{lb}(a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \frac{a^5}{5} - \&c.);$$

that is, if b be the base of a system of logarithms, this value of x will be the logarithm of $1+a$ in that system: therefore if $b = 10$ (the base of the common system), then

$$x = \frac{1}{l10}(a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \frac{a^5}{5} - \&c.);$$

or, in other words, the value of x in the equation $10^x = 1+a$ is the common logarithm of $1+a$; that is,

$$x = L(1+a) = \frac{1}{l10}(a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \&c.). \quad ----(1)$$

The factor $\frac{1}{l10} = .434294481903251827651128918916 6 = m$ (as before found) is called *the modulus* of the common system of logarithms.

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Now if we substitute $a = y$ and $\frac{1}{110} = m$ in equation (1), we have $\mathrm{L}(1+y) = m(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \text{\&c.})$. ..(L)

To find a more converging series, let $y = -z$: then we have $\mathrm{L}(1-z) = m(-z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \frac{z^5}{5} - \text{\&c.})$.

Subtracting the last equation from equation (L), and

$$\mathrm{L}(1+y) - \mathrm{L}(1-z) = m(y+z - \frac{y^2-z^2}{2} + \frac{y^3+z^3}{3} - \frac{y^4-z^4}{4} + \text{\&c.});$$

or if we make $z = y$, and recollect that $\mathrm{L}(1+y) - \mathrm{L}(1-z) = \mathrm{L}\left(\frac{1+y}{1-z}\right)$, the last equation becomes

$$\mathrm{L}\left(\frac{1+y}{1-y}\right) = 2m(y + \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} + \text{\&c.}). \quad \text{.....(L')}$$

Ex. 1. To compute the common logarithm of 2.

Let $2 = \frac{1+y}{1-y}$, whence $y = \frac{2-1}{2+1} = \frac{1}{3}$: therefore,

$$\mathrm{L}\,2 = 2m(\tfrac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7} + \text{\&c.}). \quad \text{.....(1)}$$

This is the series commonly used for computing L 2; but to find a more converging one, we may first consider $2^3 = 8 = 10-2 = 10(1-\frac{2}{10}) = 10\times\frac{8}{10} = 10\times\frac{4}{5}$; and then $\mathrm{L}\,2 = \frac{1}{3}\mathrm{L}\,2^3 = \frac{1}{3}\mathrm{L}(10\times\frac{4}{5}) = \frac{1}{3}(1 + \mathrm{L}\,\frac{4}{5})$: then make $\frac{4}{5} = \frac{1+y}{1-y}$, whence $y = \frac{4-5}{4+5} = -\frac{1}{9}$; and by formula (L'),

$$\mathrm{L}\,\tfrac{4}{5} = -2m(\tfrac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \frac{1}{7\cdot 9^7} + \text{\&c.}):$$

hence because $\mathrm{L}\,2 = \frac{1}{3}(1 + \mathrm{L}\,\frac{4}{5})$,

$$\mathrm{L}\,2 = \tfrac{1}{3}[1 - 2m(\tfrac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \frac{1}{7\cdot 9^7} + \text{\&c.})]. \quad (2)$$

Or to find a still more converging series for L 2, we might consider $2^{10} = 1024 = 1000\times\frac{1024}{1000} = 1000\times\frac{128}{125}$: then

$$\mathrm{L}\,2 = \tfrac{1}{10}\mathrm{L}\,2^{10} = \tfrac{1}{10}\mathrm{L}(1000\times\tfrac{128}{125}) = \tfrac{1}{10}(3 + \mathrm{L}\,\tfrac{128}{125}),$$

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as $\mathrm{L}\,1000 = 3$. Now employ formula (L′), and make

$$\tfrac{128}{125} = \frac{1+y}{1-y} \text{ and } y = \frac{128-125}{128+125} = \tfrac{3}{253} : \text{ consequently}$$

$$\mathrm{L}\,\tfrac{128}{125} = 2m\left(\tfrac{3}{253} + \frac{3^3}{3\cdot 253^3} + \frac{3^5}{5\cdot 253^5} + \frac{3^7}{7\cdot 253^7} + \&c.\right);$$

and therefore

$$\mathrm{L}\,2 = \tfrac{1}{10}\left[3 + 2m\left(\tfrac{3}{253} + \frac{3^3}{3\cdot 253^3} + \frac{3^5}{5\cdot 253^5} + \frac{3^7}{7\cdot 253^7} + \&c.\right)\right]. \quad (3)$$

The sum of eight or nine terms of this series gives

$\mathrm{L}\,2 = .30102999566398119521373889472449$, to 32 places.

Series (1) will require twice as many terms as (2), and four times as many as (3), to determine to the same accuracy; but all will give the same sum .3010, etc.

Ex. 2. To compute L 3, we might deliberate thus:

$$\mathrm{L}\,3 = \mathrm{L}(2\times\tfrac{3}{2}) = \mathrm{L}\,2 + \mathrm{L}\,\tfrac{3}{2};$$

but when high accuracy is required, it is better to take

$$\mathrm{L}\,3 = \tfrac{1}{4}\mathrm{L}\,3^4 = \tfrac{1}{4}\mathrm{L}\,81 = \tfrac{1}{4}\mathrm{L}\,(80\times\tfrac{81}{80}) = \tfrac{1}{4}(\mathrm{L}\,80 + \mathrm{L}\,\tfrac{81}{80})$$
$$= \tfrac{1}{4}(1 + 3\,\mathrm{L}\,2 + \mathrm{L}\,\tfrac{81}{80}),$$

as $\mathrm{L}\,80 = \mathrm{L}(10\times 2^3) = 1 + 3\,\mathrm{L}\,2$:

then by employing theorem (L′), we have $\tfrac{81}{80} = \dfrac{1+y}{1-y}$, whence

$y = \dfrac{81-80}{81+80} = \tfrac{1}{161}$; and therefore

$$\mathrm{L}\,\tfrac{81}{80} = 2m\left(\tfrac{1}{161} + \frac{1}{3\cdot 161^3} + \frac{1}{5\cdot 161^5} + \frac{1}{7\cdot 161^7} + \&c.\right) : \text{ hence}$$

$$\mathrm{L}\,3 = \tfrac{1}{4}\left[1 + 3\,\mathrm{L}\,2 + 2m\left(\tfrac{1}{161} + \frac{1}{3\cdot 161^3} + \frac{1}{5\cdot 161^5} + \frac{1}{7\cdot 161^7} + \&c.\right)\right].$$
$$= .47712125471966243729502790325511.$$

Ex. 3. To compute L 7, we might first make

$$\mathrm{L}\,7 = \mathrm{L}\,(6\times\tfrac{7}{6}) = \mathrm{L}\,6 + \mathrm{L}\,\tfrac{7}{6} = \mathrm{L}\,2 + \mathrm{L}\,3 + \mathrm{L}\,\tfrac{7}{6};$$

but it will be better to take

$$\mathrm{L}\,7 = \tfrac{1}{4}\mathrm{L}\,7^4 = \tfrac{1}{4}\mathrm{L}\,2401 = \tfrac{1}{4}\mathrm{L}\,(2400\times\tfrac{2401}{2400})$$
$$= \tfrac{1}{4}(\mathrm{L}\,2400 + \mathrm{L}\,\tfrac{2401}{2400}),$$

where $\mathrm{L}\,2400 = \mathrm{L}(100\times 2^3\times 3) = 2 + 3\,\mathrm{L}\,2 + \mathrm{L}\,3$;

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and by theorem (L′),

$$L\tfrac{2401}{2400} = 2m\left(\tfrac{1}{4801} + \frac{1}{3\cdot 4801^3} + \frac{1}{5\cdot 4801^5} + \frac{1}{7\cdot 4801^7} + \&c.\right) : \text{hence}$$

$$L7 = \tfrac{1}{4}\left[2 + 3L2 + L3 + 2m\left(\tfrac{1}{4801} + \frac{1}{3\cdot 4801^3} + \frac{1}{5\cdot 4801^5} + \&c.\right)\right].$$

The next prime numbers are 11, 13, 17, 19, 23, etc., the logarithms of which may be computed in a similar way to those of 2, 3 and 7; and the logarithms of all other numbers are easily determined from those of the primes.

DISCOVERY OF A GENERAL ROOT THEOREM FOR THE SOLUTION OF EQUATIONS OF ALL DEGREES.

ART. 6. As an introduction to this theorem, let us refer to Art. 4, equations (k) and (k′), which are

$$a = x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4} + \&c. \quad \ldots\ldots\ldots\ldots (k')$$

and

$$x = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \&c. \quad \ldots\ldots\ldots\ldots\ldots (k)$$

Both of these equations are comprehended by the equations $B = (1+a)^{\frac{1}{x}}$ and $(1+a)^{\frac{1}{x}} = (1+nx)^{\frac{1}{nx}}$; that is, when $n = 0$.

Let us prove this position by substituting the value of x as represented by equation (k), for x in equation (k′).

$$\begin{aligned}
x &= a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \&c.\\
\frac{x^2}{2} &= \quad + \frac{a^2}{2} - \frac{a^3}{2} + \frac{11a^4}{24} - \&c.\\
\frac{x^3}{2\cdot 3} &= \qquad + \frac{a^3}{6} - \frac{a^4}{4} + \&c.\\
\frac{x^4}{2\cdot 3\cdot 4} &= \qquad\qquad + \frac{a^4}{24} - \&c.\\
\&c. \qquad & \&c.;
\end{aligned}$$

which, by summing up, gives

$$\begin{aligned}
x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4} + \&c. &= a + \left(\tfrac{1}{2} - \tfrac{1}{2}\right)a^2 + \left(\tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{6}\right)a^3\\
&\quad + \left(-\tfrac{1}{4} + \tfrac{11}{24} - \tfrac{1}{4} + \tfrac{1}{24}\right)a^4 + \&c.\\
&= a; \quad \ldots\ldots\ldots\ldots\ldots\ldots (s)
\end{aligned}$$

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because $\frac{1}{2}-\frac{1}{2}=0$, $\frac{1}{3}-\frac{1}{2}+\frac{1}{6}=0$, $-\frac{1}{4}+\frac{1}{2}\frac{1}{4}-\frac{1}{4}+\frac{1}{2}\frac{1}{4}=0$, &c., whence the equation reduces to $a=a$, and proves our position correct.

Now take the equation $a = x+bx^2+cx^3+dx^4+$&c., ..(G)
and make $x = a+Aa^2+Ba^3+Ca^4+$&c.,
and then substitute this value of x in equation (G), similarly as was before done with the value of x in equation (k') : the substitution made, we have

$$\begin{aligned} x &= a + Aa^2 + Ba^3 + Ca^4 + \&c. \\ bx^2 &= \quad ba^2 + 2bAa^3 + (bA^2+2bB)a^4 + \&c. \\ cx^3 &= \quad\quad ca^3 + 3cAa^4 + \&c. \\ dx^4 &= \quad\quad\quad da^4 + \&c. \\ \&c. &\quad \&c.; \end{aligned}$$

and by summing up,

$$x+bx^2+cx^3+dx^4+\&c. = a = a+(A+b)a^2+(B+2bA+c)a^3 + (C+bA^2+2bB+3cA+d)a^4+\&c.$$

Now the coefficients of a^2, a^3, a^4, etc. must in each case be equal to zero, for the same reason as in the equation

$$a = a + (\tfrac{1}{2}-\tfrac{1}{2})a^2 + (\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{6})a^3 + \&c.$$

Hence
$$\begin{aligned} A+b = 0, \text{ or } A &= -b; \\ B+2bA+c = 0, \text{ or } B &= -2bA-c = 2b^2-c; \\ C+bA^2+2bB+3cA+d = 0, \text{ or } C &= -bA^2-2bB-3cA-d \\ &= -b^3-2b(2b^2-c)+3bc-d \\ &= -5b^3+5bc-d. \end{aligned}$$

By extending the substitution as far as the term containing gx^7 in the one and Fa^7 in the other member of the foregoing series of equalities, the values of D, E and F may likewise be found; that is, in the equation

$$x = a + Aa^2 + Ba^3 + Ca^4 + Da^5 + Ea^6 + Fa^7 + \&c.,$$

we have

$$\begin{aligned} A &= -b, \\ B &= 2b^2-c, \\ C &= -5b^3+5bc-d, \\ D &= 14b^4-21b^2c+6bd+3c^2-e, \\ E &= -42b^5+84b^3c-28b^2d-28bc^2+7be+7cd-f, \\ F &= 132b^6-330b^4c+120b^3d+180b^2c^2-36b^2e-72bcd+8bf+8ce \\ &\qquad -12c^3+4d^2-g, \\ \&c. &\quad \&c. \end{aligned}$$

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Therefore the true value of x in the equation

$$a = x + bx^2 + cx^3 + dx^4 + \&c., \quad \ldots\ldots\ldots\ldots (G)$$

by restoring values of A, B, C, etc., stands

$$\begin{aligned} x = {} & a - ba^2 + (2b^2 - c)a^3 - (5b^3 - 5bc + d)a^4 \\ & + (14b^4 - 21b^2c + 6bd + 3c^2 - e)a^5 \\ & - (42b^5 - 84b^3c + 28b^2d + 28bc^2 - 7be - 7cd + f)a^6 \\ & + (132b^6 - 330b^4c + 120b^3d + 180b^2c^2 - 36b^2e - 72bcd + 8bf \\ & \qquad + 8ce - 12c^3 + 4d^2 - g)a^7 - \&c. \quad (R) \end{aligned}$$

When the coefficient of x^2 is zero, that is, when $b = 0$, these equations (G) and (R) become

$$a = x + cx^3 + dx^4 + ex^5 + \&c. \quad \ldots\ldots\ldots\ldots (G')$$

$$x = a - ca^3 - da^4 + (3c^2 - e)a^5 + (7cd - f)a^6 + (8ce - 12c^3 + 4d^2 - g)a^7 + \&c. \quad \ldots\ldots\ldots\ldots (R')$$

It will hereafter be seen that three or four terms of equation (R) can be made to suffice in all cases, by transforming once or twice; and if only the first term a were used, and the equation transformed several times successively, it would amount to the same thing as Newton's rule of approximation.

ART. 7. As a preparatory step to applying theorem (R) to particular cases, let us consider the equation

$$a' = b'x + c'x^2 + d'x^3 + e'x^4 + \&c.,$$

whose form is different from that of

$$a = x + bx^2 + cx^3 + dx^4 + \&c.;$$

but if we divide both members of the former equation by b', it becomes

$$\frac{a'}{b'} = x + \frac{c'}{b'}x^2 + \frac{d'}{b'}x^3 + \frac{e'}{b'}x^4 + \&c.:$$

then make $a = \frac{a'}{b'}$, $b = \frac{c'}{b'}$, etc., and we get

$$a = x + bx^2 + cx^3 + dx^4 + \&c.,$$

which is equation (G), the root of which is expressed by theorem (R). Hence the first thing to be done to an equation, before applying this theorem, is to divide it throughout by the coefficient of x.

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ART. 8. Again, an equation of the form

$$a' = x^n + b'x^{n+1} + c'x^{n+2} + d'x^{n+3} + \&c.$$

must be transformed by substituting $x = r+y$; and then by dividing both members by the coefficient of y, it will be reduced to the form of equation (G).

To exemplify this, take the equation

$$4 = 2x^2 + 5x^3, \text{ and let } x = 1+y:$$

then
$$\begin{aligned} 2x^2 &= 2(1+y)^2 = 2+4y+2y^2, \\ 5x^3 &= 5(1+y)^3 = 5+15y+15y^2+5y^3; \end{aligned}$$

and therefore
$$4 = 7+19y+17y^2+5y^3,$$

or, by transposition and division by 19,

$$\frac{4-7}{19} = -\tfrac{3}{19} = y + \tfrac{17}{19}y^2 + \tfrac{5}{19}y^3.$$

This equation now has the first term of the second member such that its coefficient and exponent are both unity; and to this form every equation must be reduced, before the theorem can be applied.

ART. 9. It must also be borne in mind, that if we wish to apply theorem (R) to the equation

$$a = y + by^2 + cy^3 + dy^4 + \&c.,$$

or to the equation

$$a = z + bz^2 + cz^3 + dz^4 + \&c.,$$

then for x in the lefthand member of the equation (R) must be put y or z, or whatever letter expresses the root of the equation that is to be developed.

ART. 10. It will also be found, wherever the theorem is applied, that the least root of the equation is always the one that becomes developed; but as it is easy to transform any equation into a shape such that the least root of its derivative shall constitute a portion of the greatest or any other root of the primitive, therefore either root of any equation may be determined if we first suitably transform the given equation, and then develop the least root of this transformed equation by the theorem, and add it to the known assumed part of the root of the given equation.

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To illustrate this, take the equation

$$x^3 - 6x = 2, \quad \text{or} \quad -\tfrac{2}{6} \left(= -\tfrac{1}{3}\right) = x - \tfrac{1}{6}x^3.$$

Now in the present form the theorem will only determine the least root; but if we wish to obtain one of the other roots, we can soon find by trial that one root lies between 2 and 3, perhaps near to $\frac{5}{2}$: then let $x = \frac{5}{2}+y$ in the given equation, and we have

$$\begin{aligned} x^3 = \quad (\tfrac{5}{2}+y)^3 &= \tfrac{125}{8} + \tfrac{75}{4}y + \tfrac{15}{2}y^2 + y^3, \\ -6x = -6(\tfrac{5}{2}+y) &= -\tfrac{30}{2} - 6y : \\ \text{therefore} \qquad 2 &= \tfrac{5}{8} + \tfrac{51}{4}y + \tfrac{15}{2}y^2 + y^3, \end{aligned}$$

$$\text{or} \qquad (2-\tfrac{5}{8}) \div \tfrac{51}{4} = \frac{11}{2{\cdot}51} = y + \tfrac{30}{51}y^2 + \tfrac{4}{51}y^3.$$

The least root of this last equation, as the theorem would tell, will give the greatest value of x in the primitive equation; that is, if substituted for y in the assumption $x = \frac{5}{2}+y$.

As a further illustration, take the equation

$$x^3 - 6x^2 + 11x = 6 :$$

its three roots are 1, 2 and 3. If we make $x = 4+y$, the three roots of the transformed equation will be (as $y = x-4$)

$$1-4, \ 2-4 \text{ and } 3-4, \quad \text{or} \quad -3, \ -2 \text{ and } -1.$$

Now while the rule determines only the least root of equation

$$x^3 - 6x^2 + 11x = 6,$$

it will determine the least root of the equation whose roots are -3, -2 and -1, and thereby render known the greatest root of the given equation.

SOLUTION OF EQUATIONS OF ALL DEGREES, BY THE GENERAL APPLICATION OF THE ROOT THEOREM.

ART. 11. To have the equation

$$a = x + bx^2 + cx^3 + dx^4 + \&c. \quad \ldots\ldots\ldots\ldots (G)$$

and the corresponding theorem

$$x = a + Aa^2 + Ba^3 + Ca^4 + \&c. \quad \ldots\ldots\ldots\ldots (R)$$

express an equation of the first degree, make

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$b = 0$, $c = 0$, $d = 0$, etc. : then

$A = -b = 0$, $B = 2b^2 - c = 0$, $C = -5b^3 + 5bc - d = 0$, etc.

Hence, under this supposition, equations (G) and (R) become

$$a = x \text{ and } x = a.$$

QUADRATICS, OR EQUATIONS OF THE SECOND DEGREE.

Ex. 1. Given the equation $5 = 10x + x^2$, or $\frac{1}{2} = x + \frac{1}{10}x^2$, to to find the least value of x.

In the theorem $x = a + Aa^2 + Ba^3 + Ca^4 +$ &c., make

$$a = \tfrac{1}{2}, \quad b = \tfrac{1}{10}, \quad c = 0, \quad d = 0, \quad \text{\&c.} :$$

therefore

$$A = -b = -\tfrac{1}{10},$$

$$B = 2b^2 - c = \frac{2}{10^2} - 0 = \frac{2}{10^2},$$

$$C = -5b^3 + 5bc - d = -\frac{5}{10^3} + 0 - 0 = -\frac{5}{10^3};$$

and in like manner $D = \dfrac{14}{10^4}$, $E = -\dfrac{42}{10^5}$, $F = \dfrac{132}{10^6}$, &c.

Hence $x = \frac{1}{2} - \dfrac{1}{10 \cdot 2^2} + \dfrac{2}{10^2 \cdot 2^3} - \dfrac{5}{10^3 \cdot 2^4} + \dfrac{14}{10^4 \cdot 2^5} - \dfrac{42}{10^5 \cdot 2^6} + \dfrac{132}{10^6 \cdot 2^7} -$ &c.

$= .477225+.$

Ex. 2. Given the equation $a = x + bx^2$, to find the least value of x.

In order that the equation $a = x + bx^2 + cx^3 + dx^4 +$ &c. and the corresponding theorem $x = a + Aa^2 + Ba^3 + Ca^4 +$ &c. shall express the equation $a = x + bx^2$ and its least root, make

$$c = 0, \quad d = 0, \quad e = 0, \quad \text{\&c.} :$$

then $A = -b$, $B = 2b^2 - c = 2b^2$; and in like manner,

$C = -5b^3$, $D = 14b^4$, $E = -42b^5$, $F = 132b^6$, &c.

Hence

$$x = a - ba^2 + 2b^2a^3 - 5b^3a^4 + 14b^4a^5 - 42b^5a^6 + 132b^6a^7 - \text{\&c.} \quad \ldots(2)$$

This formula is sufficient for all quadratics, and is more convenient than the general theorem.

3. Given $54 = 365x - x^2$, or $\frac{54}{365} = x - \frac{1}{365}x^2$, to find the least value of x.

In equation (2), let $a = \frac{54}{365}$ and $b = -\frac{1}{365}$: then

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$$-ba^2 = -\left(-\frac{1}{365} \times \frac{54^2}{365^2}\right) = \frac{54^2}{365^3},$$

$$+2b^2a^3 = \frac{2}{365^2} \times \frac{54^3}{365^3} = \frac{2 \cdot 54^3}{365^5}, \quad \&c.$$

Hence $x = \frac{54}{365} + \frac{54^2}{365^3} + \frac{2 \cdot 54^3}{365^5} + \frac{5 \cdot 54^4}{365^7} + \&c. = .148005220+.$

Formula (2) is so simple in form, that the terms of the series expressing the least root can be readily written down by inspection in many cases, as in some of the following examples.

4. $x^2+x = \frac{1}{10}$:

$$x = \tfrac{1}{10} - \frac{1}{10^2} + \frac{2}{10^3} - \frac{5}{10^4} + \frac{14}{10^5} - \&c.,$$

by making $a = \frac{1}{10}$ and $b = 1$.

5. $3x^2-2x = \frac{1}{8}$, or $-\frac{1}{16} = x-\frac{3}{2}x^2$:

$$x = -\tfrac{1}{16} + \frac{3}{2 \cdot 16^2} - \frac{2 \cdot 3^2}{2^2 \cdot 16^3} + \frac{5 \cdot 3^3}{2^3 \cdot 16^4} - \&c.$$

6. $15x^2-50x = -7$, or $\frac{7}{50} = x-\frac{3}{10}x^2$:

$$x = \tfrac{7}{50} + \frac{3 \cdot 7^2}{10 \cdot 50^2} + \frac{2 \cdot 3^2 \cdot 7^3}{10^2 \cdot 50^3} + \frac{5 \cdot 3^3 \cdot 7^4}{10^3 \cdot 50^4} + \&c.$$

7. $-13 = 739x+123x^2$, or $-\frac{13}{739} = x+\frac{123}{739}x^2$:

$$x = -\tfrac{13}{739} - \frac{123 \cdot 13^2}{739^3} - \frac{2 \cdot 123^2 \cdot 13^3}{739^5} - \frac{5 \cdot 123^3 \cdot 13^4}{739^7} - \&c.$$

8. $20 = 20y+y^2$, or $1 = y+\frac{1}{20}y^2$:

$$y = 1 - \tfrac{1}{20} + \frac{2}{20^2} - \frac{5}{20^3} + \frac{14}{20^4} - \&c.$$

9. $100 = 25z+\frac{11}{259}z^2$, or $4 = z+\frac{11}{6475}z^2$:

$$z = 4 - \frac{11 \cdot 4^2}{6475} + \frac{2 \cdot 11^2 \cdot 4^3}{6475^2} - \frac{5 \cdot 11^3 \cdot 4^4}{6475^3} + \&c.$$

10. $1 = x+x^2$:

$$x = 1-1+2-5+14-42+132-\&c.$$

11. $100 = x+x^2$:

$$x = 100 - 100^2 + 2 \cdot 100^3 - 5 \cdot 100^4 + \&c.$$

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ART. 13. We see that examples 10 and 11 produce not converging series, but diverging ones; and as the results are more indefinite than the equations themselves, such equations should always be transformed once or twice, when a new equation will arise whose least root will be a small fraction. For example, make $x = r+y$ in the equation $100 = x+x^2$, and let r be a near value of x (which can be easily obtained by trial): the resulting equation, when developed, will form a rapidly converging series; and the nearer r comes to x, the swifter will be the convergence. From this it is evident that an indefinite number of series can be found, that will express the same root of an equation; a fact analogous to what we have seen in the case of logarithms, and of which further instances of confirmation will appear among the following examples.

Let us resume the equation $100 = x+x^2$, in which it is easily found by trial that x is less than 10 and greater than 9.

1°. Then let $x = 9+y$, and we have

$$\begin{aligned} x &= 9+ y, \\ x^2 &= 81+18y+y^2, \end{aligned}$$

and $$100 = 90+19y+y^2, \text{ or}$$

$$\frac{100-90}{19} = \tfrac{10}{19} = y+\tfrac{1}{19}y^2.$$

Now by the quadratic formula,

$$y = \tfrac{10}{19} - \frac{10^2}{19^3} + \frac{2\cdot 10^3}{19^5} - \frac{5\cdot 10^4}{19^7} + \&c.;$$

and as $x = 9+y$, therefore

$$x = 9 + \tfrac{10}{19} - \frac{10^2}{19^3} + \frac{2\cdot 10^3}{19^5} - \frac{5\cdot 10^4}{19^7} + \&c. \ldots (1)$$

2°. Or, we may substitute $x = 10+y$ in the given equation $100 = x+x^2$, and will then obtain $-\tfrac{10}{21} = y+\tfrac{1}{21}y^2$; whence, by developing, $y = -\tfrac{10}{21} - \frac{10^2}{21^3} - \frac{2\cdot 10^3}{21^5} - \frac{5\cdot 10^4}{21^7} - \&c.$,

and therefore

$$10+y = x = 10-\tfrac{10}{21} - \frac{10^2}{21^3} - \frac{2\cdot 10^3}{21^5} - \frac{5\cdot 10^4}{21^7} - \&c. \ldots (2)$$

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3°. A still more converging series may be found, by observing that 9.5 is a nearer value of x than either of those used in series (1) and (2): then let $x = \frac{19}{2}+y$, and the transformed equation will be $\frac{1}{4\cdot 20} = y+\frac{1}{20}y^2$; whence

$$y = \frac{1}{4\cdot 20} - \frac{1}{4^2\cdot 20^3} + \frac{2}{4^3\cdot 20^5} - \frac{5}{4^4\cdot 20^7} + \&c., \quad \text{and}$$

$$x = \tfrac{19}{2} + \frac{1}{4\cdot 20} - \frac{1}{4^2\cdot 20^3} + \frac{2}{4^3\cdot 20^5} - \frac{5}{4^4\cdot 20^7} + \&c. \quad \ldots.(3)$$

By using seven terms of series (1) or (2), $x = 9.5124922—$; and with only five terms of (3), we find $x = 9.5124921973$.

By taking successively nearer values of x, we may find other series at pleasure, each more convenient than the preceding, but perhaps none more simple than (3); yet if we make $x = 9.5125+y$, and then transform, we shall get a series that will converge twice or thrice as fast, but each term will be more complicated with large numbers.

12. Given $x^2+6x = 8$, to find one value of x.

By inspection, one value of x is a little greater than unity: then let $x = 1+y$; whence

$$x^2 = 1+2y+y^2,$$
$$6x = 6+6y,$$

and $$8 = 7+8y+y^2, \text{ or}$$

$$\frac{8-7}{8} = \tfrac{1}{8} = y+\tfrac{1}{8}y^2 : \text{ then developing,}$$

$$y = \tfrac{1}{8} - \frac{1}{8^3} + \frac{2}{8^5} - \frac{5}{8^7} + \frac{14}{8^9} - \&c., \text{ and}$$

$$x = 1 + \tfrac{1}{8} - \frac{1}{8^3} + \frac{2}{8^5} - \frac{5}{8^7} + \frac{14}{8^9} - \&c.$$

13. Given $179 = 3x+17x^2$, to find one value of x.

As x is but little greater than 3, let $x = 3+y$, and the new equation is found to be $\frac{17}{105} = y + \frac{17}{105}y^2$; whence

$$y = \tfrac{17}{105} - \frac{17^3}{105^3} + \frac{2\cdot 17^5}{105^5} - \frac{5\cdot 17^7}{105^7} + \&c.,$$

or $$x = 3 + \tfrac{17}{105} - \frac{17^3}{105^3} + \frac{2\cdot 17^5}{105^5} - \frac{5\cdot 17^7}{105^7} + \&c.$$

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Or if there be put $x = \frac{19}{6} + y$, the new equation will be $-\frac{35}{3984} = y + \frac{51}{332}y^2$: whence is found

$$x = \tfrac{19}{6} - \tfrac{35}{3984} - \frac{51\cdot 35^2}{332\cdot 3984^2} - \frac{2\cdot 51^2\cdot 35^3}{332^2\cdot 3984^3} - \&c.$$

14. Required the square root of 2, or the value of x in the equation $x^2 = 2$.

Let $x = 1 + y$; then $x^2 = 1+2y+y^2 = 2$, or $\frac{1}{2} = y+\frac{1}{2}y^2$:

then as $$y = \tfrac{1}{2} - \frac{1}{2^3} + \frac{2}{2^5} - \frac{5}{2^7} + \frac{14}{2^9} - \&c.,$$

therefore $$x = 1 + \tfrac{1}{2} - \frac{1}{2^3} + \frac{2}{2^5} - \frac{5}{2^7} + \frac{14}{2^9} - \&c. = \sqrt{2}.$$

Or if $x = 1.4 + y$, then $(1.4+y)^2 = 1.96+2.8y+y^2 = 2$, or

$$\frac{2-1.96}{2.8} = \tfrac{1}{70} = y + \tfrac{5}{14}y^2;$$ whence

$$y = \tfrac{1}{70} - \frac{5}{14\cdot 70^2} + \frac{2\cdot 5^2}{14^2\cdot 70^3} - \frac{5\cdot 5^3}{14^3\cdot 70^4} + \&c.$$: hence

$$x = \tfrac{14}{10} + \tfrac{1}{70} - \frac{5}{14\cdot 70^2} + \frac{2\cdot 5^2}{14^2\cdot 70^3} - \frac{5\cdot 5^3}{14^3\cdot 70^4} + \&c. = \sqrt{2}.$$

15. What is the square root of 3?

Answer : $$\tfrac{7}{4} - \frac{1}{7\cdot 8} - \frac{2}{7^3\cdot 8^2} - \frac{2\cdot 2^2}{7^5\cdot 8^3} - \frac{5\cdot 2^3}{7^7\cdot 8^4} - \&c.$$

ART. 14. In the preceding examples, we have only attempted to determine one of the roots in each equation, but now propose to find both. For this purpose, let the two roots of any quadratic be denoted by a' and b'; while the same, when unknown, may be indifferently represented by x. Then we shall have $x-a' = 0$ and $x-b' = 0$; and also

$$(x-a')(x-b') = x^2 - (a'+b')x + a'b' = 0:$$

therefore the sum of the roots, with the sign changed, will be equal to the coefficient of x, when the coefficient of x^2 is unity.

ART. 15. Again, a cubic equation has three roots, which may be denoted by a', b' and c': then

$$x-a' = 0, \quad x-b' = 0 \text{ and } x-c' = 0;$$

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and by multiplication,

$$(x-a')(x-b')(x-c') = x^3-(a'+b'+c')x^2+(a'b'+a'c'+b'c')x-a'b'c' = 0.$$

ART. 16. In a similar manner, if a', b', c' and d' denote the four roots of a biquadratic, we should find

$$x^4-(a'+b'+c'+d')x^3+(a'b'+a'c'+a'd'+b'c'+b'd'+c'd')x^2 \\ -(a'b'c'+a'b'd'+a'c'd'+b'c'd')x+a'b'c'd' = 0.$$

ART. 17. From the foregoing, it is concluded that any equation of the form

$$x^n+A'x^{n-1}+B'x^{n-2}+C'x^{n-3}+\&c. \text{ to (last coefficient) } x^{n-n} = 0,$$

is such, 1°, that the coefficient of the second term (A'), with its sign changed, is equal to the sum of all the roots.

2° That the coefficient of the third term (B') is equal to the sum of all the products that can be formed by every two of the roots.

3° That the coefficient of the fourth term (C'), with its sign changed, is equal to the sum of all the products that can be formed of every three of the roots; and so on to the last term.

4° That when the last term is made to represent one member of the equation (as might be done by transposition in the preceding equations), it will be equal to the product of all the roots, with its sign changed for quadratics, biquadratics, and all equations of even degree, but not changed for odd degrees.

5° Therefore, from the first conclusion, it follows that when one root of a quadratic equation is known, the other may be found by subtracting the former from the coefficient of the second term of the equation.

Let us resume example 12, which is $x^2+6x = 8$. We found

$$x = 1+\tfrac{1}{8}-\frac{1}{8^3}+\frac{2}{8^5}-\frac{5}{8^7}+\frac{14}{8^9}-\&c.:$$

then, as the sum of the roots is -6,

$$x = -7-\tfrac{1}{8}+\frac{1}{8^3}-\frac{2}{8^5}+\frac{5}{8^7}-\frac{14}{8^9}+\&c.$$

is the other root.

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Or the equation $x^2+6x = 8$ may be transformed by making $x = -7+y$, which gives $-\frac{1}{8} = y-\frac{1}{8}y^2$: whence

$$y = -\tfrac{1}{8} + \frac{1}{8^3} - \frac{2}{8^5} + \frac{5}{8^7} - \&c., \quad \text{or}$$

$$x = -7 - \tfrac{1}{8} + \frac{1}{8^3} - \frac{2}{8^5} + \frac{5}{8^7} - \frac{14}{8^9} + \&c. \text{ as before;}$$

which, subtracted from -6, gives

$$x = 1 + \tfrac{1}{8} - \frac{1}{8^3} + \frac{2}{8^5} - \frac{5}{8^7} + \frac{14}{8^9} - \&c.$$

for the other value.

Let us refer to example 13, which is $17x^2+3x = 179$, or $x^2+\frac{3}{17}x = \frac{179}{17}$. One root was found to be

$$\tfrac{19}{6} - \tfrac{3584}{3984} - \frac{51\cdot 35^2}{332\cdot 3984^2} - \frac{2\cdot 51^2\cdot 35^3}{332^2\cdot 3984^3} - \&c.:$$

then as $-\frac{3}{17}$ = the sum of the two roots, therefore

$$x = -\tfrac{3}{17} - \tfrac{19}{6} + \tfrac{3584}{3984} + \frac{51\cdot 35^2}{332\cdot 3984^2} + \frac{2\cdot 51^2\cdot 35^3}{332^2\cdot 3984^3} + \&c.$$

is the other value.

16. What are the roots of the equation $x^2-7x = -11$?

Here one value of x being near 2 and the other near 5, we may either put $x = 2+y$ or $x = 5+y$; but in one case we would obtain the lesser value of x, and in the other the greater.

Thus if $x = 2+y$, then $x^2 = \quad 4+4y+y^2$,
$\quad -7x = -14-7y$
and $\quad -11 = -10-3y+y^2$, or $\frac{1}{3} = y-\frac{1}{3}y^2$.

Then by developing,

$$y = \tfrac{1}{3} + \frac{1}{3^3} + \frac{2}{3^5} + \frac{5}{3^7} + \&c.: \text{ therefore}$$

$$x = 2 + \tfrac{1}{3} + \frac{1}{3^3} + \frac{2}{3^5} + \frac{5}{3^7} + \&c. = \text{the lesser root;}$$

and as the sum of the roots is 7, we also have

$$x = 5 - \tfrac{1}{3} - \frac{1}{3^3} - \frac{2}{3^5} - \frac{5}{3^7} - \&c. = \text{the greater root.}$$

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Or if it be desired to obtain the greater value first, then let $x = 5+y$ in the equation $x^2-7x = -11$, and we have

$$-\tfrac{1}{3} = y+\tfrac{1}{3}y^2; \text{ whence}$$

$$y = -\tfrac{1}{3} - \frac{1}{3^3} - \frac{2}{3^5} - \frac{5}{3^7} - \&c.;$$

but as $x = 5+y$, consequently

$$x = 5 - \tfrac{1}{3} - \frac{1}{3^3} - \frac{2}{3^5} - \frac{5}{3^7} - \&c. = \text{the greater root,}$$

as before.

To find a more converging series, let

$$x = (5 - \tfrac{1}{3} - \frac{1}{3^3}) + y = (5 - \tfrac{10}{27}) + y = \tfrac{125}{27} + y,$$

instead of $5+y$: this gives $-\dfrac{19}{27\cdot 61} = y + \tfrac{27}{61}y^2$;

and then developing, and introducing $x = \tfrac{125}{27} + y$, we find

$$x = \tfrac{125}{27} - \frac{19}{27\cdot 61} - \frac{19^2}{27\cdot 61^3} - \frac{2\cdot 19^3}{27\cdot 61^5} - \&c. = \text{the greater,}$$

and (as 7 = the sum of the roots),

$$x = 7 - \tfrac{125}{27} + \frac{19}{27\cdot 61} + \frac{19^2}{27\cdot 61^3} + \frac{2\cdot 19^3}{27\cdot 61^5} + \&c. = \text{the lesser}$$

root; that is, $x = 4.618034+$, or $x = 2.381966-$ *.

---

* By taking the equation $-1 = x+x^2$, and applying the quadratic formula, we get $x = -1-2-5-14-42-\&c. = -\infty$ for one, $= -\frac{1}{2}-\frac{1}{2}\sqrt{(-3)}$ by completing the square,
and $x = 2+5+14+42+\&c. = +\infty = -\frac{1}{2}+\frac{1}{2}\sqrt{(-3)}$ for the other root.
Whenever the roots are imaginary (or impossible), the sum of the series will be infinite.

Every equation of an odd degree will have at least one real root, but those of even degree may have all their roots imaginary. A mathematical question, so proposed as to contradict itself, will often produce an equation with imaginary roots, which circumstance will be shown if the sum of the series becomes infinite, whether the equation be quadratic, biquadratic, or of any higher even degree. The theorem will not determine an imaginary root, if there be any other.

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ART. 18. CUBIC EQUATIONS.

1. Given $x^3 - 3x^2 + 10x = 1$, or $\frac{1}{10} = x - \frac{3}{10}x^2 + \frac{1}{10}x^3$, to develop the least root, without transforming, by theorem (R), which is $x = a + Aa^2 + Ba^3 + Ca^4 + \&c.$

Here $a = \frac{1}{10}$, $A = -b = \frac{3}{10}$,

$$B = 2b^2 - c = 2 \times \frac{3^2}{10^2} - \frac{1}{10} = \frac{8}{10^2},$$

$$C = -5b^3 + 5bc - d = -5 \times \frac{3}{10} \times \frac{1}{10} - 0 = -\frac{15}{10^3};$$

and in like manner, $D = -\frac{456}{10^4}$, $E = -\frac{4074}{10^5}$, &c.

Hence, for the least root, we find

$$x = \frac{1}{10} + \frac{3}{10^3} + \frac{8}{10^5} - \frac{15}{10^7} - \frac{456}{10^9} - \frac{4074}{10^{11}} - \&c. = .103078.$$

2. Given $a = x + bx^2 + cx^3 + 0x^4 + 0x^5 + \&c.$, to develop x.

In the general equation

$$a = x + bx^2 + cx^3 + dx^4 + ex^5 + \&c.,$$

and the corresponding theorem

$$x = a + Aa^2 + Ba^3 + Ca^4 + Da^5 + \&c.,$$

let d, e, f, etc. be made zero : then the equations become

$$a = x + bx^2 + cx^3 + 0,$$

and (by restoring values of A, B, C, etc.),

$$x = a - ba^2 + (2b^2 - c)a^3 - (5b^3 - 5bc)a^4 + (14b^4 - 21b^2c + 3c^2)a^5 - (42b^5 - 84b^3c + 28bc^2)a^6 + (132b^6 - 330b^4c + 180b^2c^2 - 12c^3)a^7 - \&c. \quad (3)$$

This formula will answer just as well for all cubics as the general theorem, and is rather more simple, especially after the fourth term, but it becomes still further simplified when $b = 0$; that is, if, in the equation $a = x + bx^2 + cx^3$, and the corresponding formula, $b = 0$, then $a = x + cx^3$, and

$$x = a - ca^3 + 3c^2a^5 - 12c^3a^7 + \&c. \quad (3')$$

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3. Given $5 = 25x+15x^2+x^3$, or $\frac{1}{5} = x+\frac{3}{5}x^2+\frac{1}{5^2}x^3$, to find x.

Here, by formula (3), $a = \frac{1}{5}$, $b = \frac{3}{5}$, and $c = \frac{1}{5^2}$:

hence $$x = \frac{1}{5} - \frac{3}{5^3} + \frac{17}{5^5} - \frac{24}{5^6} + \&c.$$

Or, to obtain a more converging series, let $x = \frac{1}{5}+y$; or a still more converging series will be obtained by making---- $x = .18+y$. To find the decimal, $.18 = \frac{1}{5} - \frac{3}{5^3} + \frac{17}{5^5}$ nearly, is not difficult; and then transforming from the assumption $x = \frac{18}{100}+y$, will give

$$\frac{8168}{30497200} = y + \frac{155400}{304972}y^2 + \frac{10000}{304972}y^3;$$

whence, developing, etc., we obtain

$$x = \frac{18}{100} + \frac{8168}{30497200} - \frac{155400 \cdot 8168^2}{100^2 \cdot 304972^3} + \&c.$$

Two terms of this series give $x = .1802678$, so far correct.

4. Given $x^3+6x = 2$, to develop x.

First we have $\frac{2}{6} = \frac{1}{3} = x + \frac{1}{6}x^3$; and by formula (3′), we have $a = \frac{1}{3}$ and $c = \frac{1}{6}$: hence

$$x = \frac{1}{3} - \frac{1}{6 \cdot 3^3} + \frac{3}{6^2 \cdot 3^5} - \frac{12}{6^3 \cdot 3^7} + \&c. = .32748+.$$

5. Given $x^3-6x = 2$, or $-\frac{1}{3} = x - \frac{1}{6}x^3$, to develop x.

Ans. $$x = -\frac{1}{3} - \frac{1}{6 \cdot 3^3} - \frac{3}{6^2 \cdot 3^5} - \frac{12}{6^3 \cdot 3^7} - \&c. = -.33987+.$$

In the last two examples, if greater accuracy be required, make $x = .33+y$ in the former, and $x = .34+y$ in the latter: the developments of the transformed equations will then converge with great rapidity.

If it be required to determine another root of the equation $x^3-6x = 2$, it may be depressed one degree by dividing it by $x+.33987$, when the quotient will be a quadratic from which the other two roots may be obtained; but the better way will be to find by trial a near value, and then transform. By trial, we find 2 too small and 3 too large: therefore 2.5 will be

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near one root. But suppose we first transform by making

$x = 3+y$: whence we obtain $-\frac{1}{3} = y + \frac{3}{7}y^2 + \frac{1}{3\cdot 7}y^3$;

and by developing, etc., we find

$$x = 3 - \tfrac{1}{3} - \frac{3}{7\cdot 3^2} - \frac{47}{7^2\cdot 3^4} - \&c. = 2.6+.$$

Next let $x = 2.6+y = \frac{13}{5}+y$ in the equation $x^3-6x = 2$, and we shall find $\frac{1}{5\cdot 119} = y + \frac{5\cdot 13}{119}y^2 + \frac{5^2}{3\cdot 119}y^3$; and developing,

$$x = \tfrac{13}{5} + \frac{1}{5\cdot 119} - \frac{13}{5\cdot 119^3} + \frac{179}{3\cdot 119^5} - \&c. = 2.60168-.$$

The terms $\frac{13}{5}+\frac{1}{5\cdot 119} = 2.60168$, which is as near the true value of x as can be expressed without more decimals.

The third root may be found by a series similar to that of the others; but by Art. 17, 1°, the sum of all the roots is equal to minus the coefficient of x^2 in cubics, which, in the equation $x^3-6x = 2$ is zero : then if the three roots be denoted by a', b' and c', we have $a'+b'+c' = 0$; but we have found $a' = -.33987$ and $b' = 2.60168$; whence

$$c' = 0-(a'+b') = -2.60168 + .33987 = -2.26181.$$

Hence the three roots of the equation $x^3-6x = 2$ are

$$2.60168, \quad -2.26181, \quad \text{and} \quad -.33987.$$

6. Given $x^3-12x = 15$, to find the three roots of the equation.

To obtain a converging series for either of the values in the example, we must first transform. As one value is near 4, let $x = 4+y$: then

$$\begin{aligned} (4+y)^3 &= 64 + 48y + 12y^2 + y^3, \\ -12(4+y) &= -48 - 12y; \\ \text{hence} \quad 15 &= 16 + 36y + 12y^2 + y^3, \\ \text{or} \quad -\tfrac{1}{36} &= y + \tfrac{1}{3}y^2 + \tfrac{1}{36}y^3 : \\ y &= -\tfrac{1}{36} - \frac{1}{3\cdot 36^2} - \frac{7}{36^4} - \frac{5}{36^5} - \&c.; \text{ therefore} \end{aligned}$$

$$x = 4 - \tfrac{1}{36} - \frac{1}{3\cdot 36^2} - \frac{7}{36^4} - \frac{5}{36^5} - \&c. = 3.9719608 = a'.$$

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We might also find

$$x = -\tfrac{24}{10} + \tfrac{1}{220} + \frac{300}{220^3} + \frac{512500}{3\cdot 220^5} + \&c. = -2.3954260 = b',$$

by first assuming $x = -\frac{24}{10} + y$; and, finally,

$$x = c' = -(a' + b') = -1.5765348.$$

The three last examples are solved in Bonnycastle's Algebra, and some of the following will be found given as examples by different authors.

7. Given $x^3 + 10x^2 + 5x = 2600$, to find x.

First let $x = 11 + y$, as one value of x appears to be near 11 : then we find $\frac{1}{147} = y + \frac{43}{4\cdot 147}y^2 + \frac{1}{4\cdot 147}y^3$, and

$$x = 11 + \tfrac{1}{147} - \frac{43}{4\cdot 147^3} + \frac{3110}{4^2\cdot 147^5} - \&c. = 11.00679934.$$

8. Given $5x^3 + 9x^2 - 7x = 2200$, to find one value of x.

Let $x = 7 + y$, and $\frac{93}{854} = y + \frac{114}{854}y^2 + \frac{5}{854}y^3$:

$$x = 7 + \tfrac{93}{854} - \frac{114\cdot 93^2}{854^3} + \frac{2122\cdot 93^3}{854^5} - \frac{4973820\cdot 93^4}{854^7} + \&c.$$
$$= 7.1073536.$$

9. Given $5x^3 - 3x^2 - 2x = 1560$, to find one value of x.

Let $x = 7 + y$, and $\frac{6}{691} = y + \frac{96}{691}y^2 + \frac{5}{691}y^3$:

$$x = 7 + \tfrac{6}{691} - \frac{96\cdot 6^2}{691^3} + \frac{14977\cdot 6^3}{691^5} - \&c. = 7.0086719.$$

10. Given $2x^3 - 3x^2 = 3$, to find one value of x.

Let $x = 2 + y$, and $-\frac{1}{12} = y + \frac{9}{12}y^2 - \frac{2}{12}y^3$:

$$x = 2 - \tfrac{1}{12} - \frac{9}{12^3} - \frac{186}{12^5} - \&c.$$

11. What is the cube root of 2, or the value of x in the equation $x^3 = 2$?

Let $x = \frac{12}{10} + y$, and $x^3 = \frac{1728}{10^3} + \frac{432}{10^2}y + \frac{36}{10}y^2 + y^3$, or $\frac{272}{4320} = y + \frac{360}{432}y^2 + \frac{100}{432}y^3$:

$$x = \tfrac{12}{10} + \tfrac{272}{4320} - \frac{6^2\cdot 272^2}{10\cdot 434^3} + \frac{6^3\cdot 272^3}{434^5} - \&c. = \sqrt[3]{2}.$$

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A much more convenient series may be found by substituting $x = 1.26 + y$.

ART. 19. BIQUADRATIC EQUATIONS.

For the purpose of more easy reference, let us again bring to view the general equation

$$a = x + bx^2 + cx^3 + dx^4 + ex^5 + \&c., \quad \ldots\ldots\ldots\ldots (G)$$

and the corresponding theorem

$$\begin{aligned} x = {} & a - ba^2 + (2b^2 - c)a^3 - (5b^3 - 5bc + d)a^4 \\ & + (14b^4 - 21b^2c + 6bd + 3c^2 - e)a^5 \\ & - (42b^5 - 84b^3c + 28b^2d + 28bc^2 - 7be - 7cd + f)a^6 \\ & + (132b^6 - 330b^4c + 120b^3d + 180b^2c^2 - 36b^2e - 72bcd + 8bf \\ & + 8ce - 12c^3 + 4d^2 - g)a^7 - \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (R) \end{aligned}$$

This will be the most convenient formula for all equations higher than the third degree, except when some of the coefficients b, c, d, etc. of equation (G) are equal to zero.

1° If, in equation (G) and theorem (R), we make $b = 0$, we have $a = x + cx^3 + dx^4 + ex^5 + \&c.$; $\ldots\ldots\ldots (G')$

and for the root of the same,

$$\begin{aligned} x = {} & a - ca^3 - da^4 + (3c^2 - e)a^5 + (7cd - f)a^6 + (8ce - 12c^3 + 4d^2 - g)a^7 \\ & + \&c. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (R') \end{aligned}$$

2° If, in equations (G) and (R), $b = 0$, $d = 0$, $f = 0$, etc., we have $a = x + cx^3 + ex^5 + gx^7 + \&c.$; $\ldots\ldots\ldots (G'')$

and for the root of the same,

$$x = a - ca^3 + (3c^2 - e)a^5 - (12c^3 - 8ce + g)a^7 + \&c. \ldots\ldots (R'')$$

3° If, in equations (G) and (R), $c = 0$, they become

$$a = x + bx^2 + dx^4 + ex^5 + fx^6 + \&c., \quad \ldots\ldots\ldots (G''')$$

and
$$\begin{aligned} x = {} & a - ba^2 + 2b^2a^3 - (5b^3 + d)a^4 + (14b^4 + 6bd - e)a^5 \\ & - (42b^5 + 28b^2d - 7be + f)a^6 \\ & + (132b^6 + 120b^3d - 36b^2e + 8bf + 4d^2 - g)a^7 - \&c. \ (R''') \end{aligned}$$

These are all the modifications it is necessary to make, except such as will occur in practice : indeed theorem (R) might be used on all occasions, and the modifications made

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at the time of application; but formulas (R′), (R″) and (R‴), when required, will mostly dispense with such labor.

1. Given $4 = 20x+11x^2+9x^3-x^4$, or

$$\tfrac{1}{5} = x + \frac{11}{4\cdot5}x^2 + \frac{9}{4\cdot5}x^3 - \frac{1}{4\cdot5}x^4,$$

to find one value of x by theorem (R).

From the theorem, we have $a = \frac{1}{5}$,

$$b = \frac{11}{4\cdot5}, \quad c = \frac{9}{4\cdot5}, \quad \text{and } d = -\frac{1}{4\cdot5}: \text{ therefore}$$

$$x = \tfrac{1}{5} - \frac{11}{4\cdot5^3} + \frac{62}{4^2\cdot5^5} + \frac{3645}{4^3\cdot5^7} - \frac{16006}{4^4\cdot5^9} + \&c. = .1796+.$$

To find a more converging series, first transform by making $x = \frac{1}{5}+y$, or by making $x = .18+y$.

2. Given $x^4 - 5x^3 + 9x = \frac{14}{5}$, or $\frac{14}{5\cdot9} = x - \frac{5}{9}x^3 + \frac{1}{9}x^4$, to find one value of x by formula (R′).

We have $a = \frac{14}{5\cdot9}$, $c = -\frac{5}{9}$, and $d = \frac{1}{9}$, $e=0$, $f=0$, etc.:

$$x = \frac{14}{5\cdot9} + \frac{5\cdot14^3}{5^3\cdot9^4} - \frac{14^4}{5^4\cdot9^5} + \&c. = .329+.$$

Or by making $x = \frac{1}{3}+y$ in the given equation, we get

$$-\frac{11}{15\cdot202} = y - \tfrac{117}{2022}y^2 - \tfrac{99}{2022}y^3 + \tfrac{27}{2022}y^4;$$

and then, by theorem (R), we find

$$y = -\frac{11}{15\cdot202} + \frac{117\cdot11^2}{15^2\cdot202^3} - \frac{47376\cdot11^3}{15^3\cdot202^5} + \&c.,$$

and $x = \frac{1}{3} + y = .32971055.$

3. Given $x^4 - 10x = -1$, or $\frac{1}{10} = x - \frac{1}{10}x^4$, to find one value of x by formula (R′).

Observe that $a = \frac{1}{10}$, $c = 0$, $d = \frac{1}{10}$, $e = 0$, $f = 0$, etc.:

then $x = \frac{1}{10} + \frac{1}{10^5} + \frac{4}{10^9} + \&c. = .100010004.$

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4. Given $x^4-x^3+10x = 1$, or $\frac{1}{10} = x-\frac{1}{10}x^3+\frac{1}{10}x^4$, to find one value of x by formula (R′).

Ans. $x = \frac{1}{10} + \frac{1}{10^4} - \frac{1}{10^5} + \frac{3}{10^7} - \frac{7}{10^8} + \frac{52}{10^{10}} - \&c.$

By theorem (R), the four following examples are quite simple, and easily developed almost by inspection.

5. $\frac{1}{10} = x+x^2+x^3+x^4$:

$$x = \frac{1}{10} - \frac{1}{10^2} + \frac{1}{10^3} - \frac{1}{10^4} + \frac{2}{10^5} - \&c.$$

6. $-\frac{1}{10} = x+x^2+x^3+x^4$:

$$x = -\frac{1}{10} - \frac{1}{10^2} - \frac{1}{10^3} - \frac{1}{10^4} - \frac{2}{10^5} - \&c.$$

7. $\frac{1}{10} = x-x^2+x^3-x^4$:

$$x = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \frac{2}{10^5} + \&c.$$

8. $\frac{1}{10} = x-x^2-x^3-x^4$:

$$x = \frac{1}{10} + \frac{1}{10^2} + \frac{3}{10^3} + \frac{11}{10^4} + \frac{44}{10^5} + \&c.$$

9. $5 = 40x+20x^2+10x^3+5x^4$:

$$x = \frac{1}{8} - \frac{1}{2\cdot 8^2} + \frac{1}{4\cdot 8^3} - \frac{1}{8\cdot 8^4} + \frac{2}{16\cdot 8^5} - \&c.$$

10. $7 = 67x-29x^2+17x^3-11x^4$:

$$x = \frac{7}{67} + \frac{29\cdot 7^2}{67^3} + \frac{543\cdot 7^3}{67^5} + \frac{6169\cdot 7^4}{67^7} + \&c.$$

11. Given $x^4+2x^3+4x^2-8x = 8000$, to find one value of x.

Let $x = 9+y$, and $-\frac{271}{3466} = y+\frac{544}{3466}y^2+\frac{38}{3466}y^3+\frac{1}{3466}y^4$:

$$x = 9 - \frac{271}{3466} - \frac{544\cdot 271^2}{3466^3} - \frac{460164\cdot 271^3}{3466^5} - \&c. = 8.920833.$$

12. Given $x^4-5x^2 = -3$, to find one value of x.

Let $x = 1+y$, and $-\frac{1}{11} = y+\frac{9}{11}y^2+\frac{1}{11}y^3-\frac{1}{11}y^4$:

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$$x = 1 - \tfrac{1}{11} - \frac{9}{11^3} - \frac{151}{11^5} - \frac{3029}{11^7} - \&c.$$

13. What is the biquadrate root of 2, or the value of x in the equation $x^4 = 2$?

Let $x = \frac{12}{10} + y$, and

$$-\tfrac{736}{69120} = y + \tfrac{864}{69120}y^2 + \tfrac{48000}{69120}y^3 + \tfrac{10000}{69120}y^4 :$$

$$x = \tfrac{12}{10} - \tfrac{736}{69120} - \frac{86400\cdot 736^2}{69120^3} - \frac{1161216000\cdot 736^3}{69120^5} - \&c. = \sqrt[4]{2}.$$

14. Given $x^4 - 2x = 13$, to find one value of x.

Let $x = 2 + y$, and $\frac{1}{30} = y + \frac{24}{30}y^2 + \frac{8}{30}y^3 + \frac{1}{30}y^4$:

$$x = 2 + \tfrac{1}{30} - \frac{24}{30^3} + \frac{912}{30^5} - \frac{41220}{30^7} + \&c.$$

If it be required to determine another root of the equation $x^4 - 2x = 13$, it is soon found that there is a negative value near -2. Then let $x = -2 + y$, and

$$\tfrac{7}{30} = y - \tfrac{24}{30}y^2 + \tfrac{8}{30}y^3 - \tfrac{1}{30}y^4 :$$

$$x = -2 + \tfrac{7}{30} + \frac{24\cdot 7^2}{30^3} + \frac{912\cdot 7^3}{30^5} + \frac{41220\cdot 7^4}{30^7} + \&c.$$

To find a more converging series for the last root, make $x = -\frac{7}{4} + y$.

In this way, any real root of any equation, or all of them, may be expressed.

15. Given $x^4 - 3x^2 + 75x = 10000$, to find one value of x.

Let $x = 10 + y$, and $-\frac{900}{8003} = y + \frac{597}{5\cdot 803}y^2 + \frac{8}{803}y^3 + \frac{1}{5.803}y^4$:

$$x = 10 - \tfrac{900}{8003} - \frac{597\cdot 90^2}{5\cdot 803^3} - \frac{552218\cdot 90^3}{5^2\cdot 803^5} - \&c. = 9.88600+.$$

As the decimal .886 is near to $\frac{8}{9}$, then to find a more converging series, let $x = 9\frac{8}{9} + y = \frac{89}{9} + y$; or let $x = 9.886 + y$, and three terms of the series constituting the value of y will give y and x to over twenty places : one term alone will give $x = 9.88600270094$.

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ART. 20. EQUATIONS OF THE FIFTH DEGREE.

1. Given $\frac{1}{10} = x+x^2+x^3+x^4+x^5$, to find one value of x.

Employing theorem (R), we have $a = \frac{1}{10}$, $b = 1$, $c = 1$, $d = 1$, $e = 1$, $f = 0$, $g = 0$, etc.; and therefore

$$x = \tfrac{1}{10} - \frac{1}{10^2} + \frac{1}{10^3} - \frac{1}{10^4} + \frac{1}{10^5} - \frac{6}{10^7} + \&c.$$

2. $\frac{1}{10} = x+x^3+x^4+x^5$. By formula (R′),

$$x = \tfrac{1}{10} - \frac{1}{10^3} + \frac{1}{10^4} + \frac{2}{10^5} - \frac{7}{10^6} - \&c.$$

3. $\frac{1}{10} = x+x^3+x^5$. By formula (R″),

$$x = \tfrac{1}{10} - \frac{1}{10^3} + \frac{2}{10^5} - \frac{4}{10^7} + \&c.$$

4. $\frac{1}{10} = x+x^5$. By formula (R″),

$$x = \tfrac{1}{10} - \frac{1}{10^5} + \&c.$$

5. Given $x^5+2x^4+3x^3+4x^2+5x = 16$.

Let $x = 1+y$, and $\frac{1}{5\cdot7} = y + y^2 + \frac{3}{5}y^3 + \frac{1}{5}y^4 + \frac{1}{5\cdot7}y^5$: then by theorem (R), etc.,

$$x = 1 + \frac{1}{5\cdot7} - \frac{1}{5^2\cdot7^2} + \frac{7}{5^4\cdot7^3} - \frac{11}{5^5\cdot7^4} + \frac{639}{5^7\cdot7^6} - \&c.$$

6. What is the fifth root of 160000, or the value of x in the equation $x^5 = 160000$?

Let $x = 11+y$, and

$$-\tfrac{1051}{73205} = y + \tfrac{13310}{75205}y^2 + \tfrac{1210}{73205}y^3 + \tfrac{55}{73205}y^4 + \tfrac{1}{73205}y^5:$$

$$y = -\tfrac{1051}{73205} - \frac{13310\cdot1051^2}{73205^3} - \frac{265734150\cdot1051^3}{73205^5} - \&c.$$

$$x = 10.985605+ = \sqrt[5]{160000}.$$

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ART. 21. EQUATIONS OF THE SIXTH DEGREE.

1. Given $2x^6-\frac{5}{2}x^5+3x^4-\frac{3}{2}x^3+4x^2-19x = -\frac{7}{3}$, to find one value of x.

First we have

$$\frac{7}{3\cdot19} = x - \tfrac{4}{19}x^2 + \frac{3}{2\cdot19}x^3 - \frac{3}{19}x^4 + \frac{5}{2\cdot19}x^5 - \frac{2}{19}x^6 :$$

then in theorem (R), let

$$a = \frac{7}{3\cdot19},\ b = -\frac{4}{19},\ c = \frac{3}{2\cdot19},\ d = -\frac{3}{19},\ e = \frac{5}{2\cdot19},\ f = -\frac{2}{19},$$

$g = 0$, $h = 0$, etc. : therefore

$$x = \frac{7}{3\cdot19} + \frac{4\cdot7^2}{3^2\cdot19^3} + \frac{7\cdot7^3}{2\cdot3^3\cdot19^5} + \frac{833\cdot7^4}{3^4\cdot19^7} + \frac{21157\cdot7^5}{4\cdot3^5\cdot19^9} + \&c.$$

To find a more converging series, make $x = \frac{7}{3\cdot19} + y$; or, $x = \frac{1}{8}+y$ would be more simple, and give a series nearly as convergent.

2. Given $\frac{1}{10} = x - \frac{1}{2}x^3 + \frac{1}{4}x^5 - \frac{1}{6}x^6$, to find one value of x by formula (R′).

$$x = \tfrac{1}{10} + \frac{1}{2\cdot10^3} + \frac{1}{2\cdot10^5} + \frac{1}{6\cdot10^6} + \frac{1}{2\cdot10^7} + \&c.$$

ART. 22. We might multiply examples of the sixth degree, then of the seventh, and so on; but as the same theorem, or its derivatives, answer for all cases, it is unnecessary. One example, however, of the seventh degree, is added.

Find one value of x in the equation

$$\tfrac{1}{9} = x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7.$$

By theorem (R), we have

$$x = \tfrac{1}{9} - \frac{1}{9^2} + \frac{1}{9^3} - \frac{1}{9^4} + \frac{1}{9^5} - \frac{1}{9^6} + \frac{1}{9^7} - \frac{1+}{9^8} + \&c.$$

To find a more converging series, make $x = \frac{1}{9}+y$ or $\frac{1}{10}+y$.

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ART. 23. The general equation for annuities is

$$A' = \frac{p[(1+r)^n-1]}{r}, \quad \text{or } 1+\frac{A'}{p}r = (1+r)^n. \quad \ldots.(1)$$

Let us now assume $A' = 1413.98$, $p = 50$, and $n = 20$: then $1413.98 = \frac{50[(1+r)^{20}-1]}{r}$, or $28.5796 = \frac{(1+r)^{20}-1}{r}$; an equation from which it is supposed by some to be practically impossible to obtain r, except by successive approximations. The root theorem will make the approximations at once, or by one series.

If, in equation (1), we assume $\frac{A'}{p} = q$, it will stand

$$1+qr = (1+r)^n$$
$$= 1+nr+n.\frac{n-1}{2}r^2+n.\frac{n-1}{2}.\frac{n-2}{3}r^3+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}r^4+\&c. :$$

or if we transpose, and divide by $n.\frac{n-1}{2}r$, we get

$$\frac{2q-2n}{n(n-1)} = r+\frac{n-2}{3}r^2+\frac{n-2}{3}.\frac{n-3}{4}r^3+\&c.;$$

but from the above, $q = 28.2796$ and $n = 20$: therefore by substituting these values, etc., there will result

$$.043+ = r+6r^2+25.5r^3+\&c.;$$

and by the root theorem,

$$r = .043-6\times.043^2+(2\cdot6^2-25.5).043^3-\&c.$$
$$= .043-.011+.003 = .035 \text{ nearly.}$$

To find a more convergent series, let $A = .035$, and $r = A+x$ in the equation $1+qr = (1+r)^n$, and we have

$$1+Aq+qx = (1+A+x)^n;$$

or if $1+Aq = B$ and $1+A = \xi$, then

$$B+qx = (\xi+x)^n = \xi^n(1+\frac{x}{\xi})^n, \quad \text{or}$$

$$\frac{B}{\xi^n}+\frac{q}{\xi^n}x = (1+\frac{x}{\xi})^n$$
$$= 1+\frac{n}{\xi}x+\frac{n}{\xi}.\frac{n-1}{2\xi}x^2+\frac{n}{\xi}.\frac{n-1}{2\xi}.\frac{n-2}{3\xi}x^3+\&c.;$$

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or, transposing, and dividing both members by $\left(\frac{n}{\varsigma} - \frac{q}{\varsigma^n}\right)$,

$$\frac{\mathrm{B} - \varsigma^n}{n\varsigma^{n-1} - q} = x + \frac{n(n-1)}{2(n\varsigma^{n-1} - q)}\varsigma^{n-2}x^2 + \frac{n(n-1)(n-2)}{6(n\varsigma^{n-1} - q)}\varsigma^{n-3}x^3 + \&c. \quad (2)$$

Lastly, apply theorem (R), by considering

$$a = \frac{\mathrm{B} - \varsigma^n}{n\varsigma^{n-1} - q}, \quad b = \frac{n(n-1)}{2(n\varsigma^{n-1} - q)}\varsigma^{n-2}, \quad c = \frac{n(n-1)(n-2)}{6(n\varsigma^{n-1} - q)}\varsigma^{n-3}, \quad \&c.$$

Equation (2) may then stand

$$a = x + bx^2 + cx^3 + \&c., \quad \text{whose root is}$$
$$x = a - ba^2 + (2b^2 - c)a^3 - \&c.; \text{ therefore}$$
$$r = \mathrm{A} + a - ba^2 + (2b^2 - c)a^3 - \&c. \ldots\ldots\ldots\ldots\ldots (r)$$

From the above data, the numerical values of a, b, c, etc. may be determined : that of a will be a very small fraction, which will cause series (r) to converge with great rapidity.

ART. 24. CASES OF INFINITY;

OR SUCH AS WOULD BE COMPREHENDED BY THE EQUATION

$$a = x + bx^2 + cx^3 + dx^4 + \&c.,$$

when the series does not terminate.

1. Given $\frac{1}{10} = x + x^2 + x^3 + x^4 + \&c.$ without end, to find the true value of x by theorem (R).

We have $a = \frac{1}{10}$, $b = 1$, $c = 1$, $d = 1$, etc. forever; and therefore

$$x = \frac{1}{10} - \frac{1}{10^2} + \frac{1}{10^3} - \frac{1}{10^4} + \&c. = .09090909, \&c. = \frac{1}{11}.$$

2. $\frac{1}{10} = x - x^2 + x^3 - x^4 + \&c.$:

$$x = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \&c. = .11111, \&c. = \frac{1}{9}.$$

3. $\frac{1}{10} = x + x^3 + x^5 + x^7 + \&c.$:

$$x = \frac{1}{10} - \frac{1}{10^3} + \frac{2}{10^5} - \frac{5}{10^7} + \&c. = \sqrt{(26)} - 5.$$

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4. $a = x + \frac{1}{2}x^2 + \frac{1}{2\cdot3}x^3 + \frac{1}{2\cdot3\cdot4}x^4 + \&c.$:

$x = a - \frac{1}{2}a^2 + \frac{1}{3}a^3 - \frac{1}{4}a^4 + \&c. = l(1+a).$

5. $a = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \&c.$:

$x = a + \frac{1}{2}a^2 + \frac{1}{2\cdot3}a^3 + \frac{1}{2\cdot3\cdot4}a^4 + \&c.$

6. $a = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \&c.$:

$x = a - \frac{1}{3}a^3 + \frac{2}{3\cdot5}a^5 - \frac{17}{5\cdot7\cdot9}a^7 + \&c.$

7. Given the equation

$$x = \frac{1}{n-x} - \frac{1}{3(n-x)^3} + \frac{1}{5(n-x)^5} - \frac{1}{7(n-x)^7} + \&c.,$$

to find the value of x.

Make $\frac{1}{n-x} = y$; which gives $x = \frac{ny-1}{y} = n - \frac{1}{y}$, and therefore $\frac{ny-1}{y} = y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \frac{1}{7}y^7 + \&c.,$

or $ny-1 = y^2 - \frac{1}{3}y^4 + \frac{1}{5}y^6 - \frac{1}{7}y^8 + \&c.;$

and by transposition,

$$1 = ny - y^2 + \frac{1}{3}y^4 - \frac{1}{5}y^6 + \frac{1}{7}y^8 - \&c.,$$

or $$\frac{1}{n} = y - \frac{1}{n}y^2 + \frac{1}{3n}y^4 - \frac{1}{5n}y^6 + \frac{1}{7n}y^8 - \&c.$$

Here, by formula (R'''),

$a = \frac{1}{n}$, $b = -\frac{1}{n}$, $d = \frac{1}{3n}$, $e = 0$, $f = -\frac{1}{5n}$, $g = 0$, $h = \frac{1}{7n}$, etc.;

and therefore

$$y = \frac{1}{n} + \frac{1}{n}\cdot\frac{1}{n^3} + \frac{2}{n^2}\cdot\frac{1}{n^3} + \left(\frac{5}{n^3} - \frac{1}{3n}\right)\frac{1}{n^4} + \left(\frac{14}{n^4} - \frac{6}{3n^2}\right)\frac{1}{n^5}$$

$$+ \left(\frac{42}{n^5} - \frac{28}{3n^3} + \frac{1}{5n}\right)\frac{1}{n^6} + \left(\frac{132}{n^6} - \frac{120}{3n^4} + \frac{8}{5n^2} + \frac{4}{9n^2}\right)\frac{1}{n^7} + \&c.;$$

or, by simplifying,

$$y = \frac{1}{n} + \frac{1}{n^3} + \frac{5}{3n^5} + \frac{16}{5n^7} + \frac{302}{45n^9} + \&c.$$

If $n = 4.71238898$, $y = .222548$, $x = n - \frac{1}{y} = .218979.$

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$$\frac{2-3.6\,\mathrm{L}3.6}{3.6(m+\mathrm{L}\,3.6)}=\frac{5.6-7.2\,(\mathrm{L}3+\mathrm{L}2)}{3.6(m+2\,\mathrm{L}3+2\,\mathrm{L}2-1)}=-\frac{.002689}{3.6\times.990596}$$

$$=-\frac{.002715}{3.6}=a,\text{ or }ar=-.002715,\text{ and}$$

$$x=r+ar=3.6-.002715=3.597285+.$$

To determine x to a greater degree of accuracy, take the logarithms of 2 and 3 as computed heretofore to 18 places, and we may find

$$\begin{aligned}
2\,(\mathrm{L}3+\mathrm{L}2)-1=\mathrm{L}\,3.6 &= 0.556302500767287265,\\
m &= 0.434294481903251828,\\
m+\mathrm{L}\,3.6 &= 0.990596982670539093,\\
2-3.6\,\mathrm{L}3.6 &= -0.002689002762234154,\\
3.6\,(m+\mathrm{L}\,3.6) &= 3.566149137613940734,\\
a &= -0.000754035419851601,\\
m\div(m+\mathrm{L}\,3.6)=n &= 0.438416923835606983,\\
a^2 &= 0.000000568569414391,\\
n^2 &= 0.1922093991055,\\
a^3 &= -0.000000000428721477,\\
n^3 &= 0.0842678,\\
a^4 &= 0.000000000000323272.
\end{aligned}$$

Now by formula (x),

$$\begin{aligned}
a &= -0.00075403541985160,\\
-\tfrac{1}{2}n\,a^2 &= -0.00000012463522682,\\
+\left(\frac{n^2}{2}+\frac{n}{6}\right)a^3 &= -0.00000000007252862,\\
-\left(\frac{5n^3}{8}+\frac{5n^2}{12}+\frac{n}{12}\right)a^4 &= -0.00000000000005472,\\
\text{fifth term} &= -0.00000000000000004.
\end{aligned}$$

Let s = sum of five terms; then

$$\begin{aligned}
s &= -0.00075416012766180,\\
3.6s=rs &= -0.00271497645958248;
\end{aligned}$$

or, as $x=r+rs$, therefore $x=$ 3.5972850235404175, true thus far.

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